Resemblances and Disjunctions: Art, Mathematics and Economic Models[1]

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Abstract

This chapter attempts to explore possible connections between some forms of art and economic modelling. It synthesises some common insights across selected works of René Magritte and M.C. Escher. Building on them, it explores their relevance to understand certain aspects of mathematical modeling in social sciences, especially in economics. It argues for a more prominent recognition and even a celebration of the relatively unsung virtue of modeling: to highlight paradoxes and impossibilities that lie beneath various representations. Despite their subversive character, I argue that creative disjunctions can play an important epistemological role by advancing our understanding of the nature and limitations of models, at the same time opening up space for new concepts and conjectures. A case for the fruitfulness of such an approach in economics is illustrated by drawing examples from the works of Kenneth Arrow, Piero Sraffa and Vela Velupillai.

1 Introduction

The relationship between economic theory and the world of art is not immediately obvious. The former, often dubbed a dismal science, broadly deals with questions related to the creation and distribution of wealth through various modes of production, consumption and exchange. Art, on the other hand, is perceived to be the realm of the creative, expressing a wide range of ideas, skills, imagination and constantly challenging established boundaries. In the eyes of most people, the direct appeal of art to the senses, and matters of the heart and mind is perhaps far from the cold, unimaginative and calculating rationalism of economics.

One possible common ground between the two disciplines can be found in their engagement with representation. For instance, there is some level of art involved in creating a model of the economy. This sort of judiciousness involved, however, is not my direct focus here. Instead, the focus is on how works of art draw one's attention to selected aspects of what they represent or signify, which may not be obvious in the first place. Provocative artworks often shape perceptions to varying degrees by cleverly reorienting

[1] This paper is meant to be a tribute to Prof. Vela Velupillai. He always infused copious art, poetry and music in all his lectures, especially on the most technical subjects, which made them highly enjoyable for me. The panache and elegance with which he did so will forever be an elusive standard for most. A preliminary version of this paper was presented at the workshop on 'Economics & The Plastic Arts', held at Goldsmiths, University of London, during 4-5 July 2019. I thank Constantinos Repapis, Ivano Cardinale, Sarath Jakka, Diviya Pant, Astrid Van den Bossche and Yohan John for helpful discussions on related topics, even if they may not agree with the views expressed here. I also thank participants from the above workshop for useful comments and clarifications. They are not responsible for any errors, which remain my own.

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our attention, thereby stoking our imagination, creating a sense of wonder, appealing to our senses and even challenging our prevailing views of reality and the status quo.[3]

In the world of economic theory, the act of storytelling has, over the past century, increasingly moved towards employing mathematical models to represent relevant worlds or phenomena. Economic models range from those capturing aggregate macro-level phenomena such as growth, business cycles, inflation, unemployment, and international trade to those dealing with micro-level issues such as consumption choices, labour supply, and situations of bargaining. Economic theories and the worlds they capture are often mathematically characterised. Once represented, a variety of features and associated consequences are closely examined. There is a lot of literature on modeling in economics that examines a range of questions, such as the ways in which idealisations and abstractions are employed (Morgan, 2008, 2012), the nature of the worlds that models create (Sugden, 2000), their ontological status, relationship with reality, ability to explain and limitations (Lawson, 2019). Instead of attempting a detailed comparative exercise between art and economic theory as a whole, which is beyond my expertise, this chapter has a more modest goal: to find common ground between the methods and tactics employed by some artists and some mathematical economists and the way they use representation. More specifically, I focus on the theme of disjunction, manifested in the form of paradoxes, illusions, contradictions and impossibilities, both in economic theory and in art.

The reason to focus on such paradoxical results perhaps needs some justification. First, showing paradoxes and contradictions has played an important role in advancing our knowledge in virtually every field of academic inquiry. Their counter-intuitive character often draws the necessary attention of inquisitive minds and unleashes creative energies to further the investigation. They serve as effective instruments to highlight important disjunctions and issues within theories, thereby offering the necessary dose of scepticism. Paradoxes and illusions have a long history in art, and economics also has its fair share of counter-intuitive results and paradoxes (for example, Diamond-water paradox, Allais paradox, Ellsberg paradox, Edgeworth paradox).4 Although the terms paradox, contradiction and impossibility are often used interchangeably, there is a need to distinguish between these different forms of disjunctions. A paradox or a contradiction draw attention to some aspects of inquiry. They may or may not always be of interest for the larger study or theory and not all of them have important consequences. However, some of them do and these are of interest for this chapter. Impossibility claims, on the other hand, are much stronger than claims of contradictions and paradoxes. As opposed to contradictions and paradoxes, they present universal (logical) assertions about certain properties of a theoretical system under consideration. In this chapter, I do not attempt to make an exhaustive typology of different cases, instead use the term disjunction in a more general sense to refer to the notions mentioned earlier. However, the usage of the term is distinct in character, where it specifically points to epistemological limits.

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[3] Equally, knowledge of various traditions of perception that exist across different societies can influence by reorienting and redefining one's prevailing view of art. I thank Velupillai for pointing this out.

[4] I use the terms paradox, contradiction and impossibility interchangeably and there is a need to distinguish between them. A paradox or a contradiction may or may not always be of interest and not all of them have important consequences. However, some of them do and these are of interest for this chapter. Impossibility claims are much stronger than claims of contradictions and paradoxes. I use the term *disjunction* in a more general sense to refer to these different notions; however, it is distinct in character by pointing specifically to the associated epistemological limits.

Starting from the realm of art, this chapter attempts to synthesise some common insights across selected works of René Magritte and M.C. Escher (section 2), both of whom employed disjunctions, paradoxes and illusions in their art. Building on them, it explores their relevance to understanding certain aspects of mathematical modeling in social sciences, especially in economics (section 3). It argues for a more prominent recognition and even a celebration of the relatively unsung virtue of modeling: to highlight paradoxes and impossibilities that lie beneath various representations (section 4). Despite their subversive character, creative disjunctions can play an important epistemological role by advancing our understanding of the nature and limitations of models, while opening up the space for new concepts and conjectures. A case for the fruitfulness of such an approach in economics is illustrated by drawing examples from the works of Kenneth Arrow, Piero Sraffa and Vela Velupillai. Each of them, in their own way, used formal frameworks to show impossibilities, indeterminacies and undecidabilities, thereby forging new ways to think about economic problems.

Figure 1: René Magritte's The Betrayal of Images: `Ceci n'est pas une pipe', 1928-29 (oil on canvas)

2 Magritte, Escher: celebrating paradoxes

René Magritte, a celebrated Belgian surrealist artist, through his playful style raised many important questions concerning representations and the tensions that are born in transferring an idea across different mediums. Although he was a painter, his interest was in using artwork and paintings as a medium to meditate and communicate complex philosophical ideas.[5] Like many surrealists, Magritte did not like being referred to as an artist, preferring instead to be 'considered a thinker who communicated by means of paint' (Foucault, 1982: 2). Many of his paintings illustrate his keen interest in the tenuous, complex relationship between a representation and the object being represented.

More broadly, Magritte was deeply attracted to issues concerning resemblance, similitude, realism, sign and meaning; and explored these themes using disjunctions and disruptions.

[5] This outlook, which signifies 'ascendence of poetry over painting', seems to have been inspired by Giorgio de Chirico. In particular, the painting *Le chant d'amour* (1914) seems to have influenced Magritte decisively (Gablik, 1970: 25).

His famous painting, *The Treachery of Images* (1928-9, also referred to as *The Betrayal of Images* or *Ceci n'est pas une pipe)*, can be seen as a classic example of this method. Like many others in the surrealist movement, he deftly utilised the method of juxtaposing objects and concepts in a way that resulted in a paradoxical image or a situation. For instance, in the case of *The Treachery of Images*, Magritte combines a readily recognisable image of a pipe together with a legend in text that declares *Ceci n'est pas une pipe* (This is not a pipe). This seeming contradiction between the verbal and the visual is utilised as a device to highlight the distinction between the *signifier* (i.e., the sign - the image or the word) and that which is *signified*.

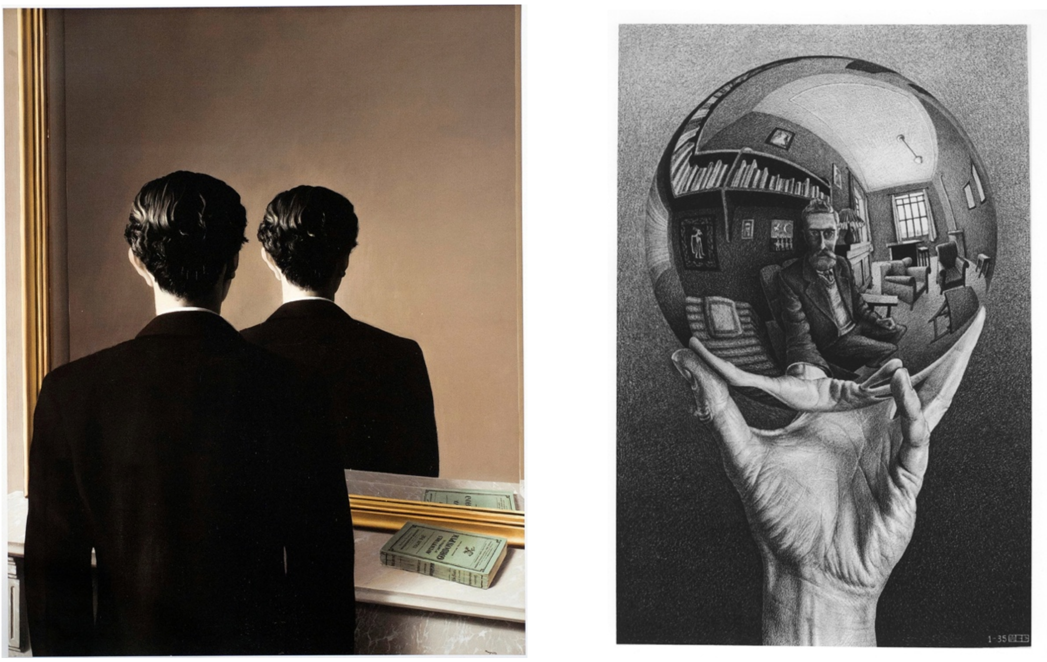

Figure 2: Recursion in Magritte and Escher. Left panel:
Rene Magritte's *Not to be Reproduced*, 1937 © ADAGP, Paris and DACS, 2021.
Right panel: M.C. Escher's "*Hand with Reflecting Sphere*"© 2021 The M.C. Escher Company-The Netherlands.
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Magritte seems to attack the uncritical equation of the essence of things with their representation, otherwise known as the object-representation divide. For Foucault, this also questions the implicit hierarchy presumed between the object and the representation, by focusing on the idea of similitude rather than that of resemblance. Similar paradoxical juxtapositions can be found in many of Magritte's paintings (*The Palace of Curtains III (1928-29), The Use of Speech (1928), The Call of the Peaks (1943), The Human Condition (1933, 1935)*). The use of seemingly inconsistent juxtapositions or disjunctions are also seen elsewhere in logic and mathematics; these carefully exploit the reference relations and the ambiguity of language, thereby creating paradoxes that pave way for a deeper inquiry into the concepts under consideration (for example, Liar's paradox, Russell's paradox).[6]

[6] A simple way to illustrate the Liar's paradox is as follows:
The following sentence is true.
The previous sentence is false.

Like Magritte, Dutch graphic artist Maurits Cornelis Escher too straddled the world of contradictions, disjunctions and paradoxes through his extremely original work. He had much in common with abstract Expressionism and Surrealism, even if he never acknowledged, actively participated or thought of himself as a member of any particular group. Escher's works, often rendered in the form of lithographs, engravings, woodcuts and sketches, exemplify the possibilities of a romantic union between art and mathematics, in particular, geometry. He never had formal training in any advanced mathematics. Yet, his artworks were pregnant with mathematical ideas and interesting intuitions.

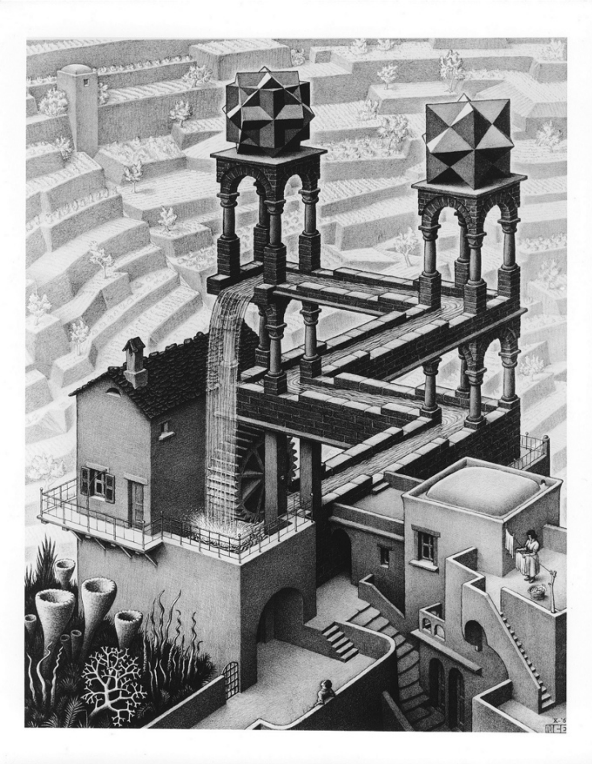

Figure 3: M.C. Escher's "Hand with Reflecting Sphere"© 2021 The M.C. Escher Company-The Netherlands. www.mcescher.com

Escher constructed impossible worlds and interesting geometries, and experimented with tessellations. However, of particular interest to him were depictions of infinity and reconciling the nature of the infinite with that of the finite. (Schattschneider, 2010: 706). Hofstadter's famous book *Gödel, Escher, Bach: an Eternal Golden Braid* uses Escher's work to develop the idea of recursion and self-reference.[7] Escher meditated a lot on the myriad of 'paradoxes of perception in representation, playing with the ambiguity of the projective systems' (Ferrero et al., 2009: 307). Several examples of these impossibilities, inconsistencies and contradictions in his work invite one to think about perception and illusion. The appealing contradictions that emanate from the representation of higher dimensional objects in two-dimensional space are noteworthy (*Drawing Hands (1948), Möbius Strip II (1963)*, for instance). Similarly, he creates seemingly plausible representations of buildings

The above example is purely in the verbal realm, exploiting disjunctive device to highlight the treacherous role played by self-reference.

[7] Magritte too can be seen playing with the idea of recursion in his painting *Not to be reproduced* (1937), which fits naturally alongside Escher's *Hand with Reflecting Sphere* (1935).

and spaces, which cleverly embed impossibilities and contradict our perceptions (e.g., *Belvedere (1958), Convex and Concave (1955), Ascending and Descending (1960)*). They become instruments or telescopes to think about and grapple with profound intuitions concerning geometry and space. Consider his work Waterfall (1961), which captures a perpetual motion that blatantly contradicts the physical laws with which we are familiar. In this piece, created with inputs from Roger and Lionel Penrose (Schattschneider, 2010: 711-12), Escher presents an object that on the face of it looks physically plausible, only to reveal the impossibilities that emerge under careful scrutiny. The use of impossible objects in art and painting has had a long history prior to Escher, dating back at least four centuries in the Netherlands. For instance, *The Magpie on the Gallows* (1568) by the Dutch painter Pieter Bruegel, features a gallow that is very much in the same class as the Penrose triangle or the impossible construction in Escher's *Waterfall*.[8]

What underlies works like *Drawing Hands* (1948) is more than the mere visual ambiguity that one finds in Wittgenstein's famous "rabbit-duck" diagram for instance. Rather, it is a kind of narrative ambiguity, whereby Escher 'ingeniously subverts of hierarchy, rendering level and meta-level perpetually reversible' (Rimmon-Kenan, 1982: 22). However, for both Magritte and Escher, creating these contradictions and impossibilities is not an attempt to delve into alternative, imaginary worlds that are completely detached from ordinary experience. On the contrary, as the philosopher Alquié points out:

> Derealization of the everyday world was always, for the surrealists, in the positive hope of possession and discovery. Surrealism is not a flight into the unreal or into dream, **but an attempt to penetrate into what has more reality than the logical and objective universe**. (Alquié, 1965: 65)

What can these impossible worlds of Escher, and the carefully constructed disjunctions of Magritte, tell us about the study of economics? Their relevance concerns the methodology of economic analysis and specifically to the part of economic theorising that is concerned with the use of models.

3 Economic models: mere resemblances?

The employment of models has been quite an important aspect of the methodology of economic analysis (both theoretical and empirical) for long, and they have acquired increasing prominence in the twentieth century. Metaphors and analogical reasoning have been routinely employed in theorizing about the economic world, and models, in their role as metaphorical and analogical devices aid the stories told by economists. In turn, empirical models have been seen as playing the role of fitting different theories to the world (more precisely to the data). Economists have tried to develop techniques that will aid them to identify the correct or true model among the many that are possible. Such models are supposed to help simplify the intricate, complex world of economic phenomena. A possible classification of economic models would be to see their roles as objects that (i) help facilitate fitting theories to the world, (ii) aid in theorising, and (iii) serve as investigative instruments (Morgan, 2008). However, the distinctions between their roles as representations, metaphor or investigative instruments may not always be distinct, and

[8] I am unsure whether Escher was directly influenced by the work of Pieter Bruegel.

this typology does not clearly define what constitutes a model and the features it embodies. In this chapter, I will focus on the use of mathematical models (not statistical or empirical models) in economic analysis, which may be seen as constituting a unique category.

Like many branches of science, economic models are closely allied to and benefit from metaphorical and analogical thinking. These metaphors range from mechanical, biological to fictional. For instance, economists speak of the: invisible hand (general equilibrium theory), rocking horses, pendulums, sunspots (business-cycle theory), predator and prey (class struggle), prisoner's dilemma, trembling hands (game theory), Robinson Crusoe, utility computers (microeconomics), turnpikes (growth theory), cob-webs, billiards players, random walks - just to name a few. Given that there is an inevitable loss of something that plagues every attempt at representation through a model, economic models are no exception. It is important to recognise and investigate the degree of sacrifice that may be involved in using them as analogue or heuristic devices. This, in turn, has important consequences for the correspondence or validity of the inferences drawn from the model to the original domain, which in this case is society.

Perhaps it is useful to briefly discuss the question of what one might mean by a model (as distinct from their use and role). Even if a precise definition may be elusive, Black (1962) provides a lucid characterisation: models are icons, analogues or abstract (symbolic) formulations that share similitude with the (real or imaginary) objects they embody. They strive to represent by preserving some properties of the original. Such invariant properties can range from relative proportions (as in scale models, e.g., miniature ships), structural properties (as in analogue models) or patterns of structural or functional relationships (as in mathematical models). François Quesnay's *Tableau économique* can be thought of as an early example of the latter class.

Theoretical models in economics predominantly belong to the class of *analogue models* more than to scale models. Therefore, desiring an exact resemblance or striving for a reproduction of all features in an economy is both impossible and unprofitable. Analogue models in economics are often mathematical, which attempt to map some selective properties of the economic system on to a mathematical or symbolic system. However, these analogue models need not exclusively be mathematical: The Phillips-Newlyn electro-mechanical hydraulic machine is a pertinent example of an analogue model that serves to capture the workings of the macroeconomic system.[9]

The choice of features to represent in a model may be guided by the structure or a part of the structure that is intended to be isomorphic with the model. Note that the notion of structure in practice can often be imaginary or a mere plausibility and not necessarily real. In a mathematical model, a social phenomenon or the structure of relationships is translated into a different medium (picture) or (symbolic) language. These models, by construction, are both simpler and more abstract than the economy.[10] One can argue that it is only by simplifying certain features, or by being unfaithful to the original in some aspects, will the model be able to serve its intended function. Thus, when looking at the

[9] This is also referred to as the Monetary National Income Analogue Computer (MONIAC). For a collection of articles that cover various aspects of MONIAC, see the special issue of the journal *Economia Politica* (Vol. XXVIII, no.1, 2011) that commemorates the 60th Anniversary of the Phillips machine.

[10] An accessible outline of the procedures concerning the use of mathematical and theoretical models can be found in Black (1962: 224-225, 230-231).

Marshallian supply-demand diagram, the criticism that it does not capture *all* that is there in a market would perhaps be misplaced. It is true that the scissor-like figure does not accurately reflect or shine a light on the myriad of complex social relationships, context, and institutional arrangements that underpin the process of exchange in an actual market. However, by virtue of being a 'model' of a market and as a heuristic fiction, it abstracts, simplifies and yet strives to map certain 'tendencies' concerning price movements. The ingenuity of the modeller then lies in the degree to which the inferences drawn from the representation can carry over to the original object of study and the level of universality that we can grant them. In this regard, it is worth noting that the kind of interpretations - whether fictitious or existential - that we employ concerning models are not trivial.

> The difference is between thinking of the electrical field *as if* it were filled with a material medium, and thinking of it *as being* such a medium. One approach uses a detached comparison reminiscent of simile and argument from analogy; the other requires an identification typical of metaphor.
>
> In *as if* thinking there is a willing suspension of ontological unbelief, and the price paid, as Maxwell insists, is absence of explanatory power. Here we might speak of the use of models as *heuristic fictions*. In risking existential statements, however, we reap the advantages of an explanation but are exposed to the dangers of self-deception by myths... (Black, 1962: 228, italics in the original)

Although these remarks were mentioned in the context of physics, they may be equally relevant for economic models, especially given the influence of physics in their development. With that brief discussion on the landscape of models in economics, we move to examine the productive role that disjunctions can play in models, methodology of economic analysis and in answering or posing ontological paradoxes.

4 Resemblances and disjunctions: in praise of subversion

The idea that models often entail simplifications, abstractions and projections on to a different medium is fairly easy to agree upon. Consequently, the possible loss of complexity and nuance involved in the original object inevitably opens up space for scepticism concerning the veracity of such representations. Based on such scepticism, one may conclude that economic models in the form of graphs, mathematical models etc. are necessarily inadequate as authentic representations. A pertinent objection at this point would be that the veracity considerations should not require that all models should serve a photograph-like function. Instead, the verisimilitude (or the lack of it) cannot be judged independent of the purpose or the intended function of the model. Such a claim would be valid, of course, only as long as the omission in representation does not deprive us of useful insights into the phenomena it intends to explain.

Another relevant point here is the intended interpretation of the model that we discussed earlier. If economic models are taken to be existential representations (*as is*), this poses a more intricate issue when these models are used for policy decisions. The possibility of feedback loops, whereby the decisions based on the model end up affecting or influencing real-world outcomes, is of real concern. Similarly, some scholars point out the possibility of economic models playing a performative role in which models go beyond merely describing an economy. Instead, these models shape the social world they are meant to be describing. This critique is well articulated in MacKenzie, Muniesa, & Siu (2007) and the vast subsequent literature it engendered.

The under-appreciated aspect of economic modelling that I wish to draw attention to is, at least in the face of it, subversive in nature. This concerns a model's ability to consciously generate *disjunctions* (i.e., showing inconsistency within the theoretical set up by highlighting paradoxes, contradictions, unsolvabilities and impossibilities) while serving as an exploratory or an investigative instrument.[11] A perception of disjunction - an apparent disconnect or contradiction between the entities involved - has often led people to declare such instances as paradoxes, impossibilities or inconsistencies. The surrealists have fruitfully employed this disjunctive device in their paintings, music, literature and theatre with admirable success. They used, *inter alia*, disjunctions to present interesting ideas in the realm of pure thought, challenge conventional perceptions and received wisdom, highlight neglected associations and advance philosophical discussions. Even though they were not economists, Magritte and Escher, regardless of their surrealist credentials, used disjunctions in *creative* ways to present contradictions.[12] They both opened up ways to challenge tacitly accepted concepts, especially those concerning the relationships between representations, perception, space, language and reality. It is this capacity of advancing new insights, concepts, hypotheses by unshackling existing perceptions and accepted conclusions that is powerful and needs to be celebrated. A modelling philosophy that focuses on identifying, constructing or deriving disjunctions in a framework or theory serves two functions: (i) to highlight aspects that are often hidden in a formal framework that sheds light on the limits of a chosen theoretical framework (ii) contradictions and attempts at explaining, in turn, set the stage for *generation* of novel hypothesis and theories, which is the main business of any scientific enterprise.

Disjunctions perceived by the reader of the model can take many forms. A non-exhaustive list could cover the following: perception of a dissonance between the features of reality and its representation in the form of a model; a disjunction between the economic intuition and the implications of the model; disjunctions that present themselves as potential surprises in the form of impossibilities, inconsistencies between aspects or notions within a model. While all of these disjunctions are legitimate, the first two are perhaps the most recognised and characterise much of the critical engagement with economic theories and models. The latter category is relatively less emphasised. In these, the disjunction often concerns the internal consistency, ambiguity, indeterminacy and even the limits of the formal language (or the kind of mathematics) in which a model is clothed.[13] For the purposes of this chapter, only the last category of disjunctions that deal with formal aspects are considered. There are several instances where this approach has been employed

[11] For instance, see Kao et al. (2012).

[12] Magritte, however, was not completely detached from the economic issues, at least from a specific political viewpoint:

> The only way that poets and painters can fight against the bourgeois economy is to give their works precisely that content which challenges the bourgeois ideological values propping up the bourgeois economy. (Magritte, 2016: xiv)

[13] Although it is worth noting that strict adherents of intuitionistic-constructive mathematics broached by L.E.J. Brouwer would demur that mathematics itself is a languageless activity and that language can only provide a description of that activity.

and has yielded interesting results. Among these, three examples in the domain of mathematical economics are worth mentioning for our present purposes.

4.1 Arrow's impossibility theorem

Arrow's impossibility theorem (Arrow , 1950, 1951) is a landmark result that concerns welfare economics, social choice theory and political science, put forward by Kenneth Arrow, the cowinner of the Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel in 1972.[14] Arrow investigated the idea of social choice (e.g. electing a prime minister, and voting for a specific policy or law as a society), ways to conceptualise it, principles to adhere to, and the possible rules or processes ('Social Welfare Function' in Arrow's terminology) by which such choices can be arrived at on the basis of preferences held by individuals in a society. Arrow's theorem showed that it was impossible to construct a mechanism or a social welfare function that was based solely on individual preference functions and at the same time satisfied a set of principles that were held as reasonable. In doing so, the impossibility theorem opened up the whole field of social choice theory and had a profound influence on the way welfare economics developed. According to Kelly (1978: 1) this 'deeply influenced theoretical welfare economics, moral and political philosophy, and mathematical approaches to microeconomic theory led to a whole new branch of welfare economics called social choice theory'. The history of this development can be found in Lützen (2019).

Arrow's impossibility theorem was one of the first impossibility results outside of mathematics and has been regarded among the more important results in the social sciences. Arrow's result also challenged the cardinal approach to welfare economics. Even though the axiomatic approach to ordered sets came into economics slightly before Arrow's work, through the work of von Neumann and Morgenstern (1947), Arrow's results drew the attention of economists to new mathematical and logical tools, which were hitherto uncommon to them. In particular, his reliance on order relations and a shift away from the use of the tools of calculus and analysis, at least in the context of welfare economics, was notable. Arrow seems to have learned about order relations and the relevant aspects of mathematical logic from the eminent mathematical logician Alfred Tarksi.[15]

To understand the context and the importance of the impossibility theorem, it is worth recalling some of the key questions in welfare economics before and around the time when Arrow was about to revolutionise the subject. Collective welfare and its maximum achievable level have traditionally been of interest to welfare economists. Cardinal approaches to welfare economics emanating from the Utilitarian tradition were dominant at the time, and focused largely on collective welfare maximisation. The social state that achieved this highest level was thought to be an ethically appropriate benchmark. There were, however, several conceptual difficulties in measuring and aggregating individual utilities to arrive at a collective measure of social welfare. A competing approach - the ordinal approach - rejected the idea of having numerical values of individual utilities and aggregating them. Instead, it focused on how different possible scenarios are ordered (e.g., scenario A is preferred to B and so on; though it remained silent about answering 'by how

[14] For a biographical article on Kenneth Arrow, see Velupillai (2019).

[15] Arrow is acknowledged in the preface of Tarski's 1941 book, *Introduction to Logic and to the Methodology of Deductive Sciences*, for his help with proofreading the English translation.

much'), and hence bypassing the issues of measurement and interpersonal comparison of utilities. Notions of the optimum in this tradition were defined without reference to cardinal values, such as the criterion by Pareto, as a potential solution. However, multiple such optimal positions were possible in principle, and some of these were ethically indefensible. The Bergson-Samuelson approach, often dubbed as the *new welfare economics*, was developed around the notion of a social welfare function (SWF). It tried to avoid cardinal measurements and tried to bridge the ordinal-cardinal divide.[16] SWF was a real-valued function, which was composed of many variables in principle; the idea was to rely on orderings as far as possible and to maximise this SWF with respect to a cardinal indicator.

Arrow completely rejected any reliance on a cardinalist approach and avoided the use of calculus and analysis, instead resorting to the machinery of order theory and logic. Instead of conceiving the social welfare function as a real-valued function that needs to be maximised using the tools of calculus, Arrow characterised preferences among different social states as weak orderings. In terms of mathematics, this indicated a shift from a continuous to a discrete analysis.

Given this background, we are now better placed to appreciate the counter-intuitive nature of the theorem. This theorem is more than just another paradoxical result concerning the theory of voting. In fact, paradoxes associated with choice and voting have a long history. For instance, the famous Condercet paradox showed that collective preferences can be cyclical or intransitive, even if the individual preferences are transitive. Different voting methods have been proposed and examined in the past by many scholars, like Nicolas de Condercet, Jean Charles de Borda, Duncan Black and others.[17] Arrow's unique contribution was in going deeper and beyond the paradoxes concerning aggregation issues and deriving a much stronger impossibility result.

Let us now move to the content of the theorem.[18] Consider x_1, x_2,..., which is taken to denote different possible social states, policies, projects or candidates among which a society has to choose from. Each individual *i*, who is a member of the society composed of N individuals, chooses an ordering relation R_i. This relation indicates their preference ranking over alternatives (say social states). Arrow departed from earlier cardinal approaches and reformulated the idea of a social welfare function as a *functional* relationship that determines an *aggregate* social ranking R on the basis of the collection of individual rankings $\{R_i\}$. The idea of rule or a process that maps individual preferences to the collective ordering of the social states is akin to a voting procedure that determines a winner from a collection of individual preferences over different candidates. The connection between voting theory and welfare economics can hence be clearly seen. The following principles or axioms can be seen as being an acceptable and reasonable requirement for a system of social choice based on democratic principles.

1. **Unrestricted domain:** for any logically possible set of individual preferences, there is a corresponding social ordering

[16] Note that the social welfare function, in this case, refers to something different from the sense in which Arrow conceives it. For a accessible history of this approach, see Lützen (2019, §3).

[17] Charles Dodgson, popularly known as Lewis Carroll, was an important contributor to this area and his work on the theory of voting was rediscovered by Duncan Black.

[18] This is a simplified exposition of the theorem to focus on the main ideas and it departs from Arrow's original version in some aspects. It closely follows Maskin and Sen (2014, pp.33-35).

2. **Pareto Principle:** If every individual prefers any x_1 to any x_2, then the society as a whole should prefer x_1 to x_2
3. **Independence of Irrelevant Alternatives:** individual rankings of x_1 and x_2 alone determine the social rankings of x_1 and x_2
4. **Nondictatorship:** There is no one distinguished voter whose preferences uniquely determine the outcome at the social level, regardless of the preferences of others.

A slightly simplified version of Arrow's impossibility theorem can be stated as follows (Maskin and Sen , 2014, p.34):19
Theorem 1 (Impossibility theorem) *If there are at least three distinct social states and a finite number of individuals, there exists no social welfare function that can simultaneously satisfy all the axioms [1]-[4] above.*

One of the implications of this elegant and powerful theorem is that these seemingly reasonable axioms are not adequate to guarantee that an aggregation procedure to determine social ordering through democratic means exists. This also alerts us to the difficulties of evaluating axioms and their plausibility in isolation, without paying due attention to other axioms with which they are being combined. Overall, Arrow's impossibility theorem highlights a crucial disjunction between individual choices and social choices under reasonable conditions, by marrying voting theory and welfare economics. This can be seen and appreciated readily through a surrealist lens as well. Despite the theorem being a negative result, it ushered impressive amounts of research in the area of social choice theory, leading to many new insights (such as the Gibbard-Satterthwaite theorem), extensions, further relaxation of Arrow's conditions and an altogether deeper understanding of inherent tensions concerning aggregations and collective choice. As Lützen (2019: 85) observes:

> As a last driving force one can mention the creative force of impossibility or paradox. Indeed, despite their negative nature, impossibility theorems rarely bar progress. On the contrary, they often result in vigorous activity. For example the discovery of incommensurability (the first rigorous impossibility proof) led the ancient Greeks to a great number of important theories. Similarly, the discovery of the unsolvability of the quintic led to the development of Galois theory and much of modern algebra. In the same vein, much of the activity in voting theory was a reaction to Condorcet's paradox, and the development of social choice theory can be (and has been) considered a response to Arrow's impossibility theorem. To circumvent the impossible has always been a strong driving force in many areas of life.

4.2 Piero Sraffa and Indeterminacy

A second example that demonstrates the inherent limitations in some economic theories is the critique developed by Piero Sraffa in his magnum opus, *Production of Commodities by Means of*

[19] Arrow originally referred to this in his book as the *General Possibility Theorem*, apparently on the advice of Tjalling Koopmans (Maskin and Sen, 2014, p.58).

Commodities (Sraffa, 1960).[20] The subtitle of the book indicates that it was a *Prelude to a Critique of Economic Theory*. Sraffa represents the economy as a production system. His choice of representation was very much in the classical and physiocratic tradition, wherein production is viewed as a circular process. Given the *methods of production* and by taking one of the distributive variables -i.e., wage or profit rate - as being exogenously given, Sraffa resolves some important issues concerning value and distribution.

Sraffa's critique is based on the idea of a self-replacing system and self-replacing prices, wages and profit rates. Self-replacing is the condition in which production can continue to take place as it did during the previous production cycle. This condition enables an exclusive focus on prices and distribution of the surplus, where produced physical quantities remain unchanged. Within this framework, Sraffa searches for those exchange-values (relative prices) between different commodities that would allow the prevailing production structure to repeat itself in the future.

Sraffa's system: a brief sketch

A very brief sketch of Sraffa's system is presented below for expository purpose.[21] Sraffa assumes an annual cycle of production and at the end of the production cycle there are n produced commodities:

$$\mathrm{b} = [b_1, b_2, \ldots b_i, \ldots, b_n]^T \quad (1)$$

i = 1,2,...,n.

The method of production producing the commodity b_i is a linear combination of means of production and labour:

$$a_i^1, a_i^2, \ldots, a_i^j, \ldots, a_i^n, l_i \rightarrow b_i \quad (2)$$

where a_i^j denotes the means of production produced by industry j used in the production of commodity i and l_i the labour used in the production of b_i.

In the case where the system produces more than the minimum amount which necessary for replacement, one ends up with a surplus that needs to be distributed. Let s_i be the surplus of commodity i available for distribution after the quantities $\{a_i^j\}$ have been put aside for the next year's production. In compact matrix notation, we have

$$\mathbf{S} = (\mathbf{B} - \mathbf{A})^T\mathbf{e} \quad (3)$$

where: e is the n×1 unit or summation vector (each element is 1); T is the transpose operator; S is the n × 1 Physical Surplus vector or Physical *Net National Product*; B is the diagonal matrix composed of gross production b as its diagonal elements. Given the knowledge of the methods of production used during the previous annual production cycle (and assuming that the same methods will be used), Sraffa solves for uniform prices that would allow the system to replicate these commodities during the next production cycles. If this is our problem setup, the accounting balance would require that:

[20] The ideas and exposition in this section draw from Zambelli (2018); Venkatachalam and Zambelli (2021,a).

[21] For a detailed version, see Zambelli (2018).

$$(\mathbf{I} + \mathbf{R})\mathbf{A}\mathbf{p} + \mathbf{L}\mathbf{w} = \mathbf{B}\mathbf{p} \quad (4)$$

where: I is the identity n × n matrix; $\mathbf{p} = [p_1, p_2, \ldots, p_n]^T$ is the price vector, $\mathbf{R} = \mathrm{diag}(r)$ is the diagonal matrix, whose diagonal elements are the rate of profits in each single industry, $r_1, r_2, \ldots, r_i, \ldots r_n$; A is the n×n matrix denotes the means of production $\{a_{ij}\}$; L is the n×1 vector whose elements $\{l_i\}$ are the labour used in production.

The system of equations in 4 is indeterminate. Sraffa simplifies the problem by assuming that the rates of profits are uniform. In that case, the n rates of profits are assumed to be equal to a single rate of profits: $r = r_1 = r_2 = \cdots = r_n$. [22] Here, there are n equations and the number of variables is reduced to n + 2 (i.e., *n* prices, a wage rate *w* and a uniform rate of profits *r*). The system is still indeterminate and it can be closed only by adding some additional exogenous elements. We can chose the physical surplus S as the *numéraire* (as Sraffa does). The economic system is described in terms of n + 1 equations (i.e., n equations in 4 and the *numéraire* equation). With the uniform rate of profits assumption, we have n + 2 variables $\{p_1, p_2, \ldots, p_n, r, w\}$. In this case, there is only one degree of freedom and one of the n + 2 variables has to be exogenously given.

Indeterminacy and the critique

Using this representation of the economy, Sraffa has shown using constructive methods that the natural prices of the commodities, in this case, are not unique and they simultaneously determine the distribution of the physical surplus. [23]Moreover, the values of gross or net output and the value of capital are themselves a function of prices and hence of distribution.

Sraffa chose a system in which the analysis relied only on quantities of goods that are used, produced and exchanged, all of which are observable in principle. He first shows that the system represented in such a manner is not determinate. There is a unique association between prices and the distribution of the physical surplus in his system. In particular, he has shown that values (i.e., self-replacing prices) and distribution cannot be computed independently of each other. Venkatachalam and Zambelli (2021) argue that this indeterminacy is at the very core of Sraffa's message. He deftly uses the classical framework to show there is a deeper indeterminacy that persists, which in turn raises uncomfortable questions for the neoclassical theory of distribution. His work constituted an important part in the famous *Cambridge Capital Controversy*, which has been regarded as one of the important debates within economic theory in the last century. This is an example of how representations cleverly geared to show indeterminacies and impossibilities - in Sraffa's case, the impossibility of determining prices and distribution, independent of each other - can pose credible challenges for conventional wisdom and foster new avenues of thinking.

It is necessary to acknowledge the pervasive temptation that exists to get rid of such indeterminacy by 'closing' the system through additional assumptions. However, such attempts miss the point and the virtue that lies beneath the choice of representations to highlight these important features. The representation provided by Sraffa is not to be mistaken for yet another system, like that of general equilibrium models that provide a mechanical

[22] Zambelli (2018) discusses this assumption. The uniform rate of profits is obviously a special case. This assumption is justified in PCMC, which is meant as a *Prelude to a Critique of Economic Theory*. [20]See Velupillai (2008) for a discussion on the constructive nature of Sraffa's arguments.

[23] See Velupillai (2008) for a discussion on the constructive nature of Sraffa's arguments.

description of how the world works. That would be a mistaken view in my opinion. The same holds for uncritically adopting Sraffa's choice of simplifying assumptions, such as the uniform rate of profits, the absence of money and so on, as defensible positions and to treat his representation as the final word. That these elements were not strictly necessary for the critique that he was developing does not automatically imply that the framework he presented has no place for these elements.[24] An awareness of the modelling philosophy advocated in this chapter, and one that Sraffa employed fruitfully, could possibly diffuse some of the fierce, but ultimately misguided, battles.

Some justification may be warranted for what has been claimed to be Sraffa's ultimate aim in his choice of representation. To this end, let us briefly consider the discussion in the correspondence between him and Pierangelo Garegnani. In 1962, the latter was preparing a review of PCMC and discussed the relation between the profit rate and the money rates of interest in his letter to Sraffa. Garegnani wonders whether the issue of the profit rate as determined by the money rates of interest is valid and provides a plausible argument. However, Sraffa was clearly wary any such mechanical ways of determining distribution within the system and states categorically that it is not the intention in his book.[25]

It is quite evident that the negation that Sraffa emphasises is the indeterminacy issue and that he is not interested in having this as a positive, mechanical model that also renders everything determinate.

> [...] in general I only wanted to send out some signals to **avoid anyone to think** that the system [of PCMC] is presented [by me] as a "foundation" for a theory of relative supplies of capital and labour! It is the <u>negation</u> that seems important to me: as to the <u>affirmative</u> I have no intention of putting forward yet **another mechanical theory** which, in one form or another, reinforces the idea that distribution is determined by natural, or technical, or perhaps even accidental circumstances, but such as to render **futile** whatever **action**, from one side or the other, aimed to **modify the distribution**. (Sraffa Papers, D3/12/111/149, translation by Stefano Zambelli, in bold emphasis added).[26]

4.3 Undecidability and uncomputability in Economic Theory

The third example, one that has been close to my own research interests, concerns the distinction (and a dissonance) between the proof of existence of an entity (concerning

[24] See Zambelli (2018) and Venkatachalam and Zambelli (2021a) on extensions of Sraffa's framework to include non-uniform rate of profits and credit and debt, respectively.

[25] For a detailed argument, see Venkatachalam and Zambelli (2021a, §3).

[26] The text in Italian:

> [...] io non ho inteso dir niente di molto impegnativo, e in generale ho solo voluto metter fuori qualche segnale **per evitare che si creda** che il sistema viene presentato come "fondamenta" per una teoria delle offerte relative di capitale e lavoro! E' la <u>negazione</u> che mi sembra importante: quanto alla <u>affermativa</u> non ho nessuna intenzione di mettere avanti **un'altra teoria meccanica** che, in una forma o nell'altra, ribadisca l'idea che la distribuzione sia determinata da circostanze naturali, o tecniche, o magari accidentali ma comunque tali da rendere **futile** qualsiasi **azione**, da una parte o dall'altra, per **modificarla** (Sraffa Papers, D3/12/111/149, in bold emphasis added).

economic models) and the ability to compute or construct these. This is relevant since not everything that has been proved to exist can be computed, sometimes even in principle, unless the choice of formalism admits only those proofs that establish existence through explicit constructions. In general, proving existence in classical mathematics does not guarantee that the mathematical object in question can be computed. This has important epistemological consequences and highlights the gulf between what we can 'prove' to exist and the possibility of computing them effectively (or decide membership). This alerts us to the importance of representations, and associated possibilities and limitations. The set of assertions that can be established as being true can depend on whether a model is expressed within a specific mathematical framework (e.g., classical mathematics or constructive mathematics), the kind of logic that underpins them, and the modes of reasoning they admit. What may be valid, admissible or true for a model in one kind of mathematics may fail to hold in another.

To understand the relevance and importance of this, it may be useful to take a brief detour into the nature of mathematics used in economic theory. To date, classical mathematics - by which I refer to set theory (Zermelo-Fraenkel variety, with the axiom of choice), real analysis and classical logic - remains the predominant kind of mathematics employed in major areas and even in the frontiers of mathematical economics. There has also been a strong reliance on the use of the axiomatic method in post-war era developments in mathematical economics. Many scholars identify these developments as having been heavily influenced by the Formalist school and the Bourbakian school of mathematics.[27] The dominant mode of theorising in this tradition has been to axiomatise an economic theory (in classical mathematics) and pose questions concerning the existence, uniqueness and stability of various forms of static or dynamic equilibria. Within this enterprise, the methods of proof and persuasion are often devoid of numerical content. Once these mathematical objects are proved to exist, these results are then utilised for applied and policy purposes by devising algorithms to compute them using digital computers.

The idea of computing is intimately connected to the notion of algorithms and it may not be possible to devise algorithms to compute all objects that are proved to exist. Computability theory or recursion theory is a branch of mathematical logic that, among other things, delineates what can and cannot be computed through a mechanical or effective procedure. Similarly, there are varieties of mathematics that may fit the bill even when we are interested in the algorithmic aspects (Martin-Löf, 1982). For instance, the mathematics of the computation is often also seen through different forms of *constructive mathematics*, which stand in contrast to classical mathematics by allowing only *constructive* proofs to establish existence.

There are important differences between these formalisms and consequently the results that we can obtain within these. For instance, the nature of the domain of numbers typically utilised in classical mathematics and analysis (real numbers) is different compared to recursion

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[27] For instance, Gerard Debreu, one of the important mathematical economists who made fundamental contributions to mathematical general equilibrium theory explicitly states (Debreu , 1984, p.395):

> After having been influenced at the Ecole Normale Superieure in the early forties by the axiomatic approach of N. Bourbaki to mathematics, I became interested in economics toward the end of World War II. ...To somebody trained in the uncompromising rigor of Bourbaki, counting equations and unknowns in the Walrasian system could not be satisfactory, and the nagging question of existence was posed. But in the late forties several essential elements of the answer were still not readily available.

Also see Velupillai (2012) on the influence of Bourbaki on mathematical economics.

theory (*rational* numbers).[28] Note that even though almost all economic quantities that we encounter are at best *rational* numbers, they are uncritically and routinely characterised as real numbers. Similarly, the notion of an algorithm or constructive procedure is central to computability theory and constructive mathematics, but they aren't central to the formalist approach to mathematics, which in turn pays very little regard to the notion of computation and numerical implementation.[29] Even such simple differences can have important consequences concerning whether the results of a model carry over across different frameworks.[30]

All of these highlight the dependencies on the choice of representation and the importance of investigating the universality (or the lack thereof) of the results obtained in mathematical economics. In other words, how immutable are the truths and insights of mathematical economic models once we change the *kind* of mathematics that is used to represent economic problems? Similarly, questions concerning the appropriateness of certain type of mathematics in relation to the economic problem at hand and the limits of purely mathematical approaches become relevant as well.

For the last four decades, Vela Velupillai has examined these important issues systematically through his research program on computable and constructive economics. Velupillai recasts many traditional economic theories, concepts and models using alternative mathematical and logical formalisms - mainly computable and constructive - and investigates the implications. Many cornerstone results that are taken for granted in mathematical economics turn out to be invalid in alternative mathematical formalisms, or that they easily cannot be translated or meaningfully computed. Velupillai (2009) presents several undecidability and uncomputability results concerning a variety of topics in mathematical economics ranging from the rational choice, general equilibrium to Nash equilibria of finite games. These results highlight the limits of specific mathematical formalisms, distinguishes the different kinds of mathematical foundations and their implications. A selection of these results (many by Velupillai, but also other scholars) are presented below (see Velupillai, 2009, Velupillai, 2010, pp.470-71):

- Nash equilibria of finite games are *constructively indeterminate*.
- Computable General Equilibria are *neither computable nor constructive*.
- The Two Fundamental Theorems of Welfare Economics are *Uncomputable and Nonconstructive*, respectively.
- There is *no effective procedure* to generate preference orderings.

[28] Computable analysis is a branch of mathematics that examines analysis from the viewpoint of computability and extends the Turing machine model to real numbers.

[29] See Mathais (1992).

[30] For instance, Bolzano-Weierstrass theorem, which is valid in real analysis and one that is often invoked either explicitly or implicitly in many mathematical economics models is not valid in constructive analysis.

- Recursive Competitive Equilibria (RCE), underpinning the Real Business Cycle (RBC) model and, hence, the Dynamic Stochastic General Equilibrium (DSGE) benchmark model of Macroeconomics, are *uncomputable*.

- There are games in which the player who in theory can always win cannot do so in practice because it is impossible to supply him with effective instructions regarding how he/she should play in order to win.[31]

- Inductively *effective* policy formalisations are impossible for complex economies.

It is evident that these results concern some of the foundational aspects of modern economic theory. Some of these results specifically highlight the presence of *undecidable disjunctions* (in the logical sense) in economic models, which cannot be made constructive. Let us briefly consider one of the results from the above list, which can help highlight the essential aspects of the narrative developed in this chapter.

Undecidability in the rational choice model

The result concerns the standard rational choice model employed in neoclassical economics. In the preference-based approach to choice, a (binary) preference relation $\mathscr{R}$ defined over a set of alternatives X is said to be *rational*, if it is complete and transitive.[32] The notion of *rational choice* is often understood to be synonymous with the idea of choosing the best preferred or maximal alternatives from a set of alternatives, on which preference orderings are generated through pairwise comparison. This model remains a cornerstone of neoclassical microeconomics and decision theory. In this framework, consistency of chosen outcomes matter more than all else and it is famously silent on the procedural aspects of decision making. Velupillai starts with standard axioms and definitions concerning choice, with an aim to unearthing the procedural aspects and make them explicit. The concept of a process is formalised in terms of an *effective procedure* or *mechanical procedure* (assuming the validity of the Church-Turing thesis) as understood in recursion theory. In doing so, he shows a formal equivalence between the rational choice behaviour of economic agents and the computational activity of a suitably programmed Turing machine.

Having established this equivalence, he further demonstrates that there is no effective procedure to generate preference ordering (i.e., there is no algorithm to systematically evaluate alternatives) by showing the choice function in question is uncomputable in general. Even if we are given a class of choice functions to generate preference orderings associated with picking maximal choices, he shows that there is no algorithm to decide whether or not an arbitrary choice function belongs to this class. In other words, the demands placed on the

[31] This is due to Michael Rabin (see, Rabin, 1957).
[32] See Mas-Colell, Whinston and Green (1995: 6).

[30]Velupillai (2000: 40-1) addresses this:

> It is quite possible that economic agents are able to rationalize their preferences using nonrecursive functions. Computable economics is, at least in part, about delineating the boundaries between the effective and noneffective process.

To what extent, then, can we firmly theorise about decision making if its procedural aspects are not accessible even to idealised forms of intuitive calculability (as captured by Church-Turing thesis)? Wittgenstein's closing remarks in the *Tractatus* may offer one answer: '*Whereof one cannot speak, thereof one must remain silent*'.

rational choice by human beings in the neoclassical model of choice surpasses the power of even the ideal computing machines. This immediately begs the question of its relevance in understanding actual human behaviour and forces one to think about relaxing the requirements we place on rational behaviour. Note that a demonstration of non-effectivity of generation of preference orderings does not imply that agents do not have or use preferences. On the contrary, it highlights the limitations of the representation in question.[30]

But our interest is in the methodological aspects of showing uncomputability and undecidability. By recasting the same questions using a different kind of mathematical framework (in his case, recursion theory) and thus bringing together two different worlds, Velupillai points us to the limitations of a specific formalisation and prompts us to think in new ways.

Velupillai's results, like those of Arrow's, may seem subversive or negative, but only at the outset. A careful examination of these opens up new vistas of thinking about economic theorising and it has given rise to a research program on computable and constructive approaches to economic theorizing. More interestingly, it points us to the dissonance between the pseudoprecisions of the mathematical models on the one hand and the 'conceptual fragility of the economic underpinnings' (Velupillai, 2015) on the other. Similar tussles based on disjunctions in mathematical philosophy, where attempts to reconcile formal mathematics and intuitive notions have resulted in deep debates that have clarified the scope and limits of mathematical knowledge (Horsten and Welch, 2016).

5 Conclusion

Magritte often indulged in a deliberate method to obscure things from the observer, 'forcing us to question and think for ourselves about the world' (Day, 2018). Escher experimented with infinities, recursions and employed carefully constructed deceptions in his geometric objects that make us question about the nature of space and reality. The examples discussed in the domain of economics in this chapter point to remarkable similarities in terms of the meta *methods* employed by these artists and the economists.

Velupillai's style of demonstrating tensions that arise between different choice of underlying mathematics – i.e., medium of representation - bears a lot of similarity to the way in which Magritte demonstrates the tension between the visual and verbal mediums in *The Treachery of Images*. Sraffa's attempts to showcase indeterminacies and the inescapable simultaneity (of prices and distribution) inherent in some theoretical models of the economy can be fruitfully interpreted through the techniques employed by Escher in his *Hand with a reflecting sphere*. Likewise, the device employed by Arrow to highlight the emergence of impossibilities from seemingly plausible or standard premises are similar to the way Escher demonstrates (architectural) impossibilities in an otherwise normal or innocuous setting in the *Waterfall*.

That said, there sure are important differences between them and that these similarities cannot be stretched too far beyond reason. Nor do these individuals seem to have had any explicit influence on each other concerning the method discussed here. However, they all share a common ground in questioning representations, employing disjunctions creatively to show certain limitations. Through these examples discussed above, I hope to have persuaded the reader that there is at least a case to see these attempts in economic modelling as sharing similarities in character between themselves and with these artists. In my opinion, they all echo the spirit in which Magritte claimed that de Chirico's painting *Le chant d'amour* (1914) showcased the 'ascendance of poetry over painting'. There is much that economists can learn from various forms of art, and I believe that it can help them understand their own quests and methods better.

I conclude with a succinct reflection by Velupillai on the approach fostered by his mentor, the economist Richard Goodwin, who was also an artist:

> To be skeptical is an art - but it can be fostered, paradoxically, by the formalism of a mathematics of ambiguity, entirely based on the Erlanger Program. It was, in art, fostered by Escher, trained in geometry by Coxeter.
>
> ...*I believe geometry can be harnessed to teach and represent economic indeterminacy and ambiguity in fruitful ways*. This was why Goodwin was aware of the need for rigorous approximations of the conceptual bases of economic theory. He recognized that **our concepts are always an approximation to a reality, the appearance of which was always deceptive, never completely encapsulable in any formalism** - that was something he had to grapple with as a painter, every time he put canvas on easel, and, then, brush to canvas.
>
> (Velupillai , 2015: 1494, emphasis added.)

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