Self-replacing prices with credit and debt

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Abstract

Sraffa advanced his critique of economic theory (Sraffa, 1960) based on the notion of a production system that replicates itself. Prices associated with exchanges in such a system ought to be such that producers and workers would have the necessary purchasing power. The original Sraffian schemes do not include the possibility for exchanges to take place through deferred means of payments. We generalise this system by introducing the possibility of deferred means of payments. We characterise the the domain of self-replacing prices in the presence of credit, debt and non-uniform rate of profits. We provide a constructive procedure to compute the relevant self-replacing prices.
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1 Introduction

What should be the exchange-values (relative prices) between different commodities in a market such that the prevailing production structure can repeat itself in the future? This question was posed by Piero Sraffa in his book (Sraffa, 1960, PCMC, henceforth). He developed system in which production was seen as a circular process in the classical and physiocratic vein. In his original system, there was no credit and debt (deferred means of payments) or money. Given that Sraffa had paid considerable attention to monetary issues in his early writings (Sraffa, Sraffa, 1922a,b, 1932), this absence may seem conspicuous. Furthermore, considering the ubiquitous role that money and credit play in the capitalistic production processes, potential implications of this omission may pique one's curiosity further (Venkatachalam and Zambelli, 2021).

The idea of a self-replacing system - where production can continue to take place as it did during the previous production cycle - and the associated notions of self-replacing prices, wages and profit rates were utilised in PCMC. This idea is important because it allows one to focus on prices and distribution of the surplus in a situation where the physical quantities produced remain unchanged. The central message of Sraffa's (prelude to a) critique, as argued in Venkatachalam and Zambelli (2021), is the insufficiency of economic mechanisms or market forces to exclusively determine values (prices, profit rates and wage rates) independent of distribution. Their simultaneous determination and, consequently, the overall *indeterminacy* in the system are crucial.

Money may not have been strictly necessary for Sraffa's critique of economic theory, however the possibility of extending his system to incorporate money has not been explored in sufficient detail.[1] It is also unclear *apriori* whether the thrust of his important critique will remain valid under such extensions. Even if money was inessential to the structure devised by Sraffa given its intended purpose, this does not imply that his theoretical apparatus has no place for some form of money.

Recently, Zambelli has extended the original Sraffian system by relaxing the assumption of uniform rate of profit (Zambelli, 2018b). In this paper, we extend the system by introducing the possibility of exchanging real goods (or services) against deferred means of payments (money, credit and debt). The role of credit and debt becomes relevant once we consider that prices which may not originally allow for self-replacing may do so once

[1] There have been previous attempts in the literature (Panico, 1988; Pivetti, 1991; Ciccarone, 1998) to introduce money and credit inside the Sraffian setup. However, to the best of our knowledge, a convincing generalisation of Sraffian schemes to incorporate credit and debt have not yet been accomplished.

deferred means of payment are introduced. They allow the system to be self-replacing by facilitating an appropriate transfer of purchasing power between agents.

Following Sraffa's method of investigation, we search for the set of prices that will allow the economic system to self-replicate when deferred means of payments are introduced. Note that these relative prices are to be distinguished from the actual market prices. The former is to be understood within the context of thought experiments, aimed at studying conditions that allow the economic system to repeat a previously observed production cycle. We call these exchange-values *self-replacing prices.*[2] As with PCMC, in this paper we do not provide a theory of prices or distribution. For given past methods of production, output and labour usage, our objective is to determine the set of prices, wage rate and the possible financial conditions that would allow self-reproduction.

We show that the introduction of credit and debt enlarges the domain of prices that allow the system to replicate the existing production structure, thus creating the necessary flexibility and structural viability within the system. We provide an explicit characterisation of the prices and wage rates that allow self-replacement, for a given distribution. The framework developed is general enough to accommodate differential or non-uniform rates of profits across industries. Section 2 presents Sraffian schemes and characterises the self-replacing prices when there are no deferred means of payments. Section 3 systematically introduces credit and debt into this system and derives explicit expressions for computing self-replacing prices and wage rates. In section 4, we demonstrate these ideas through example from PCMC and employ computational methods to identify the relevant domain of prices, wage/profit rates, aggregate value of capital associated with self-replacement. We also study the impact of financial interest rates on the time taken to repay the debts. Section 5 concludes and outlines avenues for further generalisations. We show that the important elements of Sraffa's critique concerning indeterminacy remains unaffected in the presence of credit, debt and non-uniform rates of profit.

2 Sraffian schemes: a mathematical representation

2.1 Production, consumption and distribution.

2.1.1 Production

As in PCMC (p.3, §1 and p.10, §9), we assume that there is *'an annual cycle of production with an annual market'*. At the end of the production cycle there are n produced commodities, $b_i, i = 1, 2 \ldots, n$. The *method of production* (PCMC, p.3, §1 and p. 6 §4) associated with each commodity b_i is a linear combination of means of production and labour:

$$a_i^1, a_i^2, \ldots, a_i^j, \ldots, a_i^n, \ell_i \rightarrow b_i \tag{2.1}$$

where a_i^j denotes the means of production produced by industry j used in the production of commodity i and ℓ_i the labour used in the production of b_i.

The analysis begins with a privileged observer at the beginning of the *market day* after the harvest, who can observe (i) methods of production used during the previous production cycle (say, a year), (ii) quantity of labour used and available, and (iii) the actual quantity of produced goods $[b_1, b_2, \ldots, b_n]^T$ *brought* to the market.

The system is said to be self-replacing if, at the end of the market day and after the exchanges, the producers would have the necessary means of production at their disposal to replicate the entire production process of the previous year.

[2]These prices can be seen as bookkeeping values to ensure appropriate allocation of purchasing power among the producers and the workers necessary for the system to be in a self-replacing condition.

$$b_i \xrightarrow{exchange} \overbrace{a_i^1, a_i^2, \ldots, a_i^j, \ldots a_i^{n-1}, a_i^n, \ell_i}^{\textit{Market Day}} \xrightarrow{production} b_i \tag{2.2}$$

For the whole system, this circular flow of production can be summarized as below:

$$\begin{array}{ccccccccc}
b_1 & \xrightarrow{exchange} & a_1^1 & \ldots & a_1^j & \ldots & a_1^n & \ell_1 & \xrightarrow{production} & b_1 \\
b_2 & \xrightarrow{exchange} & a_2^1 & \ldots & a_2^j & \ldots & a_2^n & \ell_2 & \xrightarrow{production} & b_2 \\
\vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \xrightarrow{production} & \vdots \\
b_i & \xrightarrow{exchange} & a_i^1 & \ldots & a_i^j & \ldots & a_i^n & \ell_i & \xrightarrow{production} & b_i \\
\vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \xrightarrow{production} & \vdots \\
b_n & \xrightarrow{exchange} & a_n^1 & \ldots & a_n^j & \ldots & a_n^n & \ell_n & \xrightarrow{production} & b_n
\end{array} \tag{2.3}$$

In compact matrix notation, this can be expressed as:

$$\mathbf{b} \xrightarrow{exchange} \mathbf{A}, \boldsymbol{\ell} \xrightarrow{production} \mathbf{b} \tag{2.4}$$

where: $\mathbf{A}$ is an $n \times n$ matrix whose entries are the amount of means of production used $\{a_i^j\}$; $\boldsymbol{\ell}$ is a $n \times 1$ vector whose elements $\{\ell_i\}$ indicate the quantity of labour used in production; $\mathbf{b}$ is $n \times 1$ vector whose elements $\{b_i\}$ indicate the amount of good i harvested.[3]

2.1.2 Surplus for distribution or consumption

The economy under consideration may produce more than what is the minimum necessary for replacement and in this case, there is a surplus available that is to be consumed or distributed (PCMC, p.6, §4). The physical surplus generated by the system during the production period is:

$$\begin{array}{ccccc}
s_1 & = & b_1 & - & \sum_{i=1}^n a_i^1 = b_1 - \mathbf{e}^T \mathbf{a}^1 \\
s_2 & = & b_2 & - & \sum_{i=1}^n a_i^2 = b_2 - \mathbf{e}^T \mathbf{a}^2 \\
\vdots & = & \vdots & - & \vdots \\
s_n & = & b_n & - & \sum_{i=1}^n a_i^n = b_n - \mathbf{e}^T \mathbf{a}^n
\end{array} \tag{2.5}$$

where s_i is the surplus of commodity i available for distribution after the necessary means of production required by all industries have been set aside for the next production cycle. Alternatively, this can be seen as the quantity produced in the previous period that remains once the inputs used in production by all industries have been removed. In compact matrix notation, we have

$$\mathbf{s} = (\mathbf{B} - \mathbf{A})^T \mathbf{e} \tag{2.6}$$

where: $\mathbf{e}$ is the $n \times 1$ unit or summation vector; T is the transpose operator; $\mathbf{s}$ is the $n \times 1$ physical surplus vector (or physical *Net National Product*); $\mathbf{B}$ is the $n \times n$ diagonal matrix, with gross production $\mathbf{b}$ as its diagonal elements. Clearly, self-replacing condition implies that the surplus is exactly equal to the consumption vector, i.e., *net* investment is zero.

[3] In the sequel, bold lowercase letters (e.g., $\boldsymbol{\ell}$, $\mathbf{s}$) signify row or column vectors and matrices will be represented using bold uppercase letters (e.g., $\mathbf{A}$). Row i and column j of a matrix $\mathbf{A}$, would be written as $\mathbf{a}_i$ and $\mathbf{a}^j$, respectively.

2.2 Self–replacing prices: a motivation

Following Sraffa, we assume that the wage is paid *post factum* (PCMC, 9-10, §8-9). Given the knowledge of the methods of production used in the previous production cycle (and assuming these do not change), Sraffa searches for uniform prices and wages that would allow the system to replicate. These prices, following the classical tradition, have been referred to by different names (PCMC, p. 7-8, §8). We refer to them as *self-replacing prices*. The rationale and interpretation of these prices is the following:

> *The significance of the equations is simply this: that if a man fell from the moon on the earth, and noted the amount of things consumed in each factory and the amount produced by each factory during a year, he would deduce at which values the commodities must be sold, [. . .] and the process of production repeated.*
>
> *Sraffa(1927 or 1928, D3/12/7, emphasis added)*[4]

These prices ought to be such that the exchange process during the market day would result in: $\mathbf{b} \xrightarrow{exchange} \mathbf{A}, \boldsymbol{\ell}, \mathbf{s}$. Once the means of production (commodities and labour) necessary for the next production period are bought, the surplus $\mathbf{s}$ is what remains available for consumption by the owners of the industries and labourers. In the next production period, $\mathbf{A}$ and the labour $\boldsymbol{\ell}$ are transformed into final output: $\mathbf{A}, \boldsymbol{\ell} \xrightarrow{production} \mathbf{b}$.

Given an arbitrary, candidate vector of commodity prices $\bar{\mathbf{p}}$ and wage rate $\bar{w}$, which are exogenously given, we have the following inequalities:

$$
\begin{array}{ccc}
\overbrace{a_1^1\bar{p}_1 + \ldots + a_1^j\bar{p}_j + \ldots + a_1^n\bar{p}_n + \ell_1\bar{w}}^{\substack{\textit{Hypothetical}\\ \textit{Expenditures}}} & \lesseqgtr & \overbrace{b_1\bar{p}_1}^{\substack{\textit{Hypothetical}\\ \textit{Revenues}}} \\
a_2^1\bar{p}_1 + \ldots + a_2^j\bar{p}_j + \ldots + a_2^n\bar{p}_n + \ell_2\bar{w} & \lesseqgtr & b_2\bar{p}_2 \\
\vdots + \vdots + \vdots + \vdots + \vdots + \vdots & \lesseqgtr & \vdots \\
a_i^1\bar{p}_1 + \ldots + a_i^j\bar{p}_j + \ldots + a_i^n\bar{p}_n + \ell_i\bar{w} & \lesseqgtr & b_i\bar{p}_i \\
\vdots + \vdots + \vdots + \vdots + \vdots + \vdots & \lesseqgtr & \vdots \\
a_n^1\bar{p}_1 + \ldots + a_1^j\bar{p}_j + \ldots + a_n^n\bar{p}_n + \ell_n\bar{w} & \lesseqgtr & b_n\bar{p}_n
\end{array}
\tag{2.7}
$$

In matrix notation, this can be written as:

$$
\overbrace{\mathbf{A}\bar{\mathbf{p}} + \boldsymbol{\ell}\bar{w}}^{\substack{\textit{Hypothetical}\\ \textit{Expenditures}}} \quad \lesseqgtr \quad \overbrace{\mathbf{B}\bar{\mathbf{p}}}^{\substack{\textit{Hypothetical}\\ \textit{Revenues}}}
\tag{2.8}
$$

By definition, the difference between revenues and expenditures would correspond to

[4]Catalogue of Sraffa Papers, Wren Library, Trinity College, Cambridge, edited by Jonathan Smith. It is dated between 1927 and 1928. See Gilibert (2006, p.28) and Gilibert (2003). From the quotation above we have removed the phrase *([...])* *" if the rate of interest must be uniform"*. This is because for the purpose of the current paper, the uniform rate of profits condition is treated as an assumption. Though this assumption may have been important for Sraffa's critique of economic theory, it is not strictly necessary while utilising Sraffa's schemes to build more general systems. See Zambelli (2018b) for a comprehensive treatment of the case with non-uniform rate of profits.

the profit levels of the industries. Consequently, the following equation should hold[5]:

$$(\mathbf{I}+\bar{\mathbf{R}})\mathbf{A}\bar{\mathbf{p}}+\boldsymbol{\ell}\bar{w}=\mathbf{B}\bar{\mathbf{p}} \tag{2.9}$$

where: $\mathbf{I}_{n\times n}$ is the identity matrix; $\bar{\mathbf{R}}=diag(\bar{\mathbf{r}})$ is the diagonal matrix, whose diagonal elements are the rate of profits in each single industry, $\bar{r}_1,\bar{r}_2,\dots\bar{r}_n$; $\mathbf{A_{n\times n}}$ is the matrix denoting the means of production $\{a_i^j\}$; $\boldsymbol{\ell}_{n\times 1}$ is the vector whose elements $\{\ell_i\}$ indicate the amount of labour used in production.[6] We use the bar symbol on top of the variables $(\bar{\mathbf{p}},\bar{w},\bar{\mathbf{r}})$ to clearly distinguish that these are candidate values and not necessarily associated with self-replacement.[7]

The left and right hand sides of eqs. 2.7 and 2.8 indicate hypothetical or bookkeeping expenditures (or costs of production) and revenues, respectively, of individual industries. We emphasise that candidate prices, self-replacing prices and market prices are three distinct notions. Candidate prices are arbitrary prices in our thought-experiment. Self-replacing prices are those prices that ensure that self-reproduction of the system. Unlike market prices, they need not necessarily be those exchange prices which are realised in the market. If $\bar{\mathbf{p}}$ were the actual exchange prices for which left-hand side is greater than the right-hand side of eq. 2.8, then there is at least one industry for which the bookkeeping expenditures (left-hand side) would be greater that of the revenues (right-hand side). In this case, the rate of profits of that industry will be negative.

2.3 Computation of self-replacing prices

We will explicitly characterise the self-replacing prices and wage rate in this subsection. The value of the surplus $\mathbf{s}^T\mathbf{p}$ is distributed among n industries and workers as below:

- $d_i\mathbf{s}^T\mathbf{p}$ is the share going to industry i, where $i=1,2,\dots n$
- $d_W\mathbf{s}^T\mathbf{p}$ is the share that goes to the workers

For a system with surplus, we have that $\sum_1^{n+1} d_z=\sum_1^{n} d_i+d_W=1$, where z is the index for the entries of the distribution vector, which includes both producers and workers.

$$\mathbf{d}\mathbf{s}^T\mathbf{p}=\overbrace{\begin{bmatrix} d_1\mathbf{s}^T\mathbf{p}\\ d_2\mathbf{s}^T\mathbf{p}\\ \vdots\\ d_n\mathbf{s}^T\mathbf{p}\\ d_W\mathbf{s}^T\mathbf{p}\end{bmatrix}}^{\substack{\textit{Distribution of}\\ \textit{Surplus or NNP}}}=\overbrace{\begin{bmatrix} b_1p_1-\mathbf{a}_1\mathbf{p}-\ell_1 w\\ b_2p_2-\mathbf{a}_2\mathbf{p}-\ell_2 w\\ \vdots\\ b_np_n-\mathbf{a}_n\mathbf{p}-\ell_n w\\ \mathbf{e}^T\boldsymbol{\ell}w\end{bmatrix}}^{\substack{\textit{Purchasing Capacity}\\ \textit{Income}}}=\overbrace{\begin{bmatrix} r_1\mathbf{a}_1\mathbf{p}\\ r_2\mathbf{a}_2\mathbf{p}\\ \vdots\\ r_n\mathbf{a}_n\mathbf{p}\\ \mathbf{e}^T\boldsymbol{\ell}w\end{bmatrix}}^{\substack{\textit{Purchasing Capacity}\\ \textit{Income}}} \tag{2.10}$$

[5]The *accounting* implicit in eq. 2.9 is consistent with the choice made in Sraffa (1960). We believe that an alternative option to compute profit rates by also including labour costs would be simpler and more appropriate. Nevertheless the qualitative conclusions would not change. The difference would be that the eq. 2.9 would have to be rewritten as $(\mathbf{I}+\mathbf{R})(\mathbf{A}\mathbf{p}+\boldsymbol{\ell}w)=\mathbf{B}\mathbf{p}$.

[6]Eq. 2.9 is a system of n equations with $2n+1$ variables: n prices $\mathbf{p}$; n profit rates $\mathbf{r}$; the wage rate, w. With the addition of a *numéraire*, the number of equations is $n+1$ and if the prices are given the number of unknowns reduces to $n+1$, i.e, the profit rates $\mathbf{r}$ and the wage rate w can be uniquely determined.

[7]For these prices to be self-replacing prices, an additional sufficiency condition needs to be satisfied.

Formally, the value of the Net National Product (Y^{NNP}) is expressed as:

$$Y^{NNP} = \mathbf{s}^T\mathbf{p} \tag{2.11}$$

In value terms, the value of the surplus that goes to producers and workers can be expressed as:[8]

$$Y^{NNP} = \mathbf{s}^T\mathbf{p} = \overbrace{\mathbf{e}_{n\times1}^T\mathbf{RAp}}^{\substack{\textit{Share to}\\ \textit{Producers (value)}}} + \overbrace{\mathbf{e}_{n\times1}^T w\boldsymbol{\ell}}^{\substack{\textit{Share to}\\ \textit{Workers (value)}}} \tag{2.12}$$

The prices are typically expressed in terms of a chosen *numéraire*. This *numéraire* can be a single or a composite commodity.[9] We pick the surplus $\mathbf{s}$ among the possible composite commodities to be set as a *numéraire*.[10] When $\mathbf{s}^T\mathbf{p} = 1$, the vector $\mathbf{d} = [d_1, d_2, \ldots, d_n, d_W]^T$ denotes distribution of the surplus $\mathbf{s}$ both in value and physical terms. Another advantage of our chosen *numéraire* is that the wage rate w could be directly interpreted as the share of the physical surplus that goes to workers.

The above accounting identity in eq. 2.10 can now be written in a more compact form as:

$$\mathbf{d} = \begin{bmatrix} \mathbf{d}_{n\times1} \\ d_W \end{bmatrix} = \begin{bmatrix} (\mathbf{B}-\mathbf{A}) & -\boldsymbol{\ell} \\ \mathbf{0}_{1\times n} & \mathbf{e}^T\boldsymbol{\ell} \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ w \end{bmatrix} = \begin{bmatrix} \mathbf{RA} & \mathbf{0}_{n\times1} \\ \mathbf{0}_{1\times n} & \mathbf{e}^T\boldsymbol{\ell} \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ w \end{bmatrix} \tag{2.13}$$

By rearranging, we have:

$$\begin{bmatrix} \mathbf{p} \\ w \end{bmatrix} = \begin{bmatrix} (\mathbf{B}-\mathbf{A}) & -\boldsymbol{\ell} \\ \mathbf{0}_{1\times n} & \mathbf{e}^T\boldsymbol{\ell} \end{bmatrix}^{-1} \mathbf{d} = \begin{bmatrix} (\mathbf{RA})^{-1} & \mathbf{0}_{n\times1} \\ \mathbf{0}_{1\times n} & (\mathbf{e}^T\boldsymbol{\ell})^{-1} \end{bmatrix} \mathbf{d} \tag{2.14}$$

The vector $\mathbf{d}$ is the physical distribution of the surplus $\mathbf{s}$ to producers and workers, but it is also the distribution in value terms, i.e., the share in value of the net national product shown in eq.(2.12). These equations enable us to find the set of prices that allow the system to self-replace itself. When the distribution $\mathbf{d}$ is given, prices $\mathbf{p}$ and the wage rate w are determined. From eq.(2.14), we see that distribution and prices are determined *simultaneously*.

3 Exchanges and deferred means of payments

Traditional Sraffian schemes outlined in the earlier section have not included the possibility of having deferred means of payments, i.e., credit or debt. In reality, these deferred

[8] $\mathbf{e}_{n\times1}$ is the summation vector. For the special case in which the rate of profits is uniform, r, as in PCMC, we have:

$$\mathbf{s}^T\mathbf{p} = \mathbf{e}_{n\times1}^T r\mathbf{Ap} + \mathbf{e}_{n\times1}^T w\boldsymbol{\ell}$$

[9] A composite commodity is a vector, say $\boldsymbol{\eta}$. Prices have to be such that $\boldsymbol{\eta}^T\mathbf{p} = 1$.

[10] Note that this was also the initial *numéraire* chosen by Sraffa (PCMC, p.11). It is convenient to measure prices, $\mathbf{p}$ in terms of the purchasing power of the Physical Surplus or Physical Net National Product. Once we pick the Surplus $\mathbf{s}$ as the *numéraire*, by definition the following relation should hold:

$$\mathbf{s}^T\mathbf{p} = \mathbf{e}^T(\mathbf{B}-\mathbf{A})\mathbf{p} = 1$$

This simplifies the analysis without changing the substance of the argument. Relative price ratios do not change with a change in the *numéraire*.

means of payments are an important characteristic of the capitalistic mode of production (Venkatachalam and Zambelli, 2021). Therefore, we generalise the framework with the possibility of actors in the system to transfer their purchasing power through having deferred means of payments.

These payments can be viewed as a form of an I Owe You (IOU). They can include all forms of financial contracts, where there is an enforceable promise to return goods in the future. These obligations may be specified with an explicit delivery date (as in the case of forward contracts associated with real goods), or with a relatively loose delivery date (as in the case of standard means of exchange such as cash, checks, debt and credit accounts, bonds etc.).

Given that commodities can be exchanged with a promise to pay back at a future point in time, one can ask interesting questions and perform thought-experiments. We can explore the set of prices and possible financial conditions that would allow the system to reproduce itself. In other words, this amounts to identifying the set of self-replacing prices.

In the following section, we incorporate deferred means of payments inside the traditional Sraffian schemes. We present a taxonomy of exchanges that are possible within this economic system. We systematically introduce credit and debt, ensuring accounting balance in each period. We arrive at the general expressions to determine the self-replacing prices, associated with a given distribution of the surplus and financial balances of actors.

3.1 A taxonomy of exchanges

Following Irving Fisher, the types of exchanges can be classified into three groups[11]:

i) **Barter exchanges** (or equivalents): the exchange of goods against goods, i.e., those who buy commodities are also selling other commodities in their possession in exchange. So far as means of payments are used only temporarily (i.e., not deferred outside the accounting period), these exchanges can be seen as barter equivalents within the same period. There are no changes in the financial positions of the participants due to these barter exchanges.

ii) **Exchange of commodities with credit and debt**: the exchange of IOUs against goods. This occurs because the buyers and sellers have the possibility of conducting an exchange since they are willing to accept future promises to pay. The entity selling this commodity would see their financial assets increase and, correspondingly, their counterpart would have their financial liabilities increase. This might take place by the writing off of means of payments previously generated or by issuing new means of payments. Furthermore, this could take place through an institution, like the banking system, or through direct contracts. Unlike barter (or equivalents), in this case there are changes in the financial positions of the participants from one period to another;

iii) **Pure financial exchanges**: the exchange of IOUs against each other. These involve exchanges only among financial contracts or promises to pay with real goods or services at a deferred point in time. There are many types of financial contracts

[11] The precise quote is the following:

> ...we may classify exchanges into three groups: the exchange of goods against goods, or barter; the exchange of money against money, or changing money; and the exchange of money against goods, or purchase and sale (Fisher, 1911, p.13).

that can be generated and exchanged. These contracts might change the future obligations, however, unlike i) or ii) as defined above, they do not involve actual buying and selling of real goods within the accounting period.

In this paper, we consider only exchanges type (i) and (ii).

3.2 Barter exchanges

We start by considering barter exchanges (or equivalents) that take place within a given accounting period. We can partition the economy into two groups: industries and workers. In the case of barter, for each industry i, the revenues accrue from the sale of its produce b_i. Similarly for workers, the revenues accrue in the form of wages paid for their labour. We can summarise them as below:

$$\begin{aligned} &\textit{Industry(i): Barter Revenues} &&= b_i^{Barter} p_i \qquad i = 1, \ldots, n \\ &\textit{Workers: Barter Revenues} &&= \mathbf{e}^T \boldsymbol{\ell}^{Barter} w \end{aligned} \tag{3.1}$$

For each industry i, expenditures under barter arise from the outlay for means of production $\mathbf{a_i}$, labour ℓ_i and final consumption by the industrial sector $\mathbf{c_i}$. For workers, the expenditures occur while purchasing consumption, $\mathbf{c_W}$.

$$\begin{aligned} &\textit{Industry(i): Barter Expenditures} &&= \mathbf{a}_i^{Barter} \mathbf{p} + \ell_i^{Barter} w + \mathbf{c}_i^{Barter} \mathbf{p} \quad i = 1, \ldots, n \\ &\textit{Workers: Barter Expenditures} &&= \mathbf{c}_{\mathbf{W}}^{Barter} \mathbf{p} \end{aligned} \tag{3.2}$$

For given prices and wages, the accounting for the whole system is constrained as follows:

$$\begin{bmatrix} \mathbf{B}^{Barter} & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & \mathbf{e}^T \boldsymbol{\ell}^{Barter} \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ w \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{Barter} & \boldsymbol{\ell}^{Barter} \\ \mathbf{0}_{1\times n} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ w \end{bmatrix} + \mathbf{C}^{Barter} \mathbf{p} \tag{3.3}$$

where $\mathbf{B}^{Barter}$ refers to the physical quantities actually sold in exchange for means of production ($\mathbf{A}^{Barter}$), labour ($\boldsymbol{\ell}^{Barter}$) and consumption ($\mathbf{C}^{Barter}$). Note that the consumption term, $\mathbf{C}^{Barter}$, includes consumption by industries as well as workers.[12] When the prices are not self-replacing prices, we have that $\mathbf{B}^{Barter} < \mathbf{B}$, $\mathbf{A}^{Barter} < \mathbf{A}$, $\boldsymbol{\ell}^{Barter} < \boldsymbol{\ell}$.

3.3 Exchange of commodities with credit and debt

Before introducing deferred means of payments into our framework, we need to understand the conditions that may necessitate them. Let us consider eqn. (2.8) and the possibility that the hypothetical expenditures do not equal the hypothetical revenues for an arbitrary price vector $\bar{\mathbf{p}}$ and wage rate $\bar{w}$. We will have industries for whom the hypothetical expenditures would be higher than the hypothetical revenues. Consequently, they would

[12] $\mathbf{C}^{Barter}$ has $n+1$ rows and n columns. The first n rows correspond to consumption by owners of n industries and the $n+1$ row refers to the consumption by the workers. Given the self-replacing assumption, all surplus is consumed in the system and there is no net investment. This implies that $\mathbf{s} = \left(\mathbf{C}^{Barter}\right)^T \mathbf{e}$.

not have the required purchasing power to buy the necessary means to replicate the production. There will be unsold goods in the system.

However, even under these hypothetical prices, there is a possibility for goods to be exchanged for a promise to pay at a later date. We now introduce the possibility of having deferred means of payments: industries lacking purchasing power would be able to obtain the necessary means of production by agreeing to a deferred payment in the future. Similarly, the industries in potential 'financial surplus' would be able to sell all of their product by agreeing to deferred payments by the borrowers.

3.3.1 A virtual bank as clearing house

For an economy to be in a state of self-replacement, we would require that the quantities $\mathbf{A}$, $\mathbf{b}$, $\boldsymbol{\ell}$, $\mathbf{s}$ have to be restored. For this to be achieved, the inequalities of eq. 2.7 would have to be *eliminated*. As we outlined, this is possible only if we allow for the possibility of deferred payments, which might take the form of *IOUs*. Note that IOUs may be generated and transferred from one period to another, which in turn might influence the set of self-replacing prices in the subsequent periods.

Producers in different industries can sell or buy commodities either (i) in return for other physical commodities or (ii) in exchange for IOUs. The same would apply for workers. For instance, they could sell a part of their labour in exchange for current payments in terms of produced commodities and another part in exchange for future promises to pay by the employer. Whether these exchanges involving future payments take place bilaterally or through a clearing house is not of consequence at this point. For simplicity, we assume that there is a virtual bank or clearing house: a central bank which ensures a smooth functioning of the payment system.

For both industries and workers, the revenues and expenditures are composed of two components, Barter and Credit-Debt:

$$\textit{Revenue} = \textit{Barter Revenue} + \Delta \textit{Credit} \tag{3.4}$$

$$\textit{Expenditure} = \textit{Barter Expenditure} + \Delta \textit{Debt} \tag{3.5}$$

In general, we are interested in finding conditions (not just prices) that would allow the economic system to be in a self-replacing state in the circular process of production extending into the future. In our framework, for each industry i and workers ($\mathbf{w}$), the revenues are the following:

$$\begin{aligned} \textit{Industry(i): Barter Revenues} &= b_i p_i \\ &= b_i^{Barter} p_i + b_i^{Credit} p_i \\ \textit{Workers: Barter Revenues} &= \mathbf{e}^T \boldsymbol{\ell} w \\ &= \mathbf{e}^T \boldsymbol{\ell}^{Barter} w + \mathbf{e}^T \boldsymbol{\ell}^{Credit} w \end{aligned} \tag{3.6}$$

Correspondingly, the expenditures can be written as:

$$
\begin{aligned}
\textit{Industry(i): Barter Expenditures} &= \mathbf{a}_i\mathbf{p} + \ell_i w + \mathbf{c}_i\mathbf{p} \\
&= \mathbf{a}_i^{Barter}\mathbf{p} + \ell_i^{Barter} w + \mathbf{c}_i^{Barter}\mathbf{p} + \\
&\quad \mathbf{a}_i^{Debt}\mathbf{p} + \ell_i^{Debt} w + \mathbf{c}_i^{Debt}\mathbf{p}
\end{aligned} \tag{3.7}
$$

$$
\begin{aligned}
\textit{Workers: Barter Expenditures} &= \mathbf{c}_{\mathbf{W}}\mathbf{p} \\
&= \mathbf{c}_{\mathbf{W}}^{Barter}\mathbf{p} + \mathbf{c}_{\mathbf{W}}^{Debt}\mathbf{p}
\end{aligned}
$$

In matrix notation, we can rewrite this as:

$$
\begin{bmatrix} \mathbf{B}^{Barter} & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & \mathbf{e}^T\boldsymbol{\ell}^{Barter} \end{bmatrix}
\begin{bmatrix} \mathbf{p} \\ w \end{bmatrix}
+
\begin{bmatrix} \mathbf{B}^{Credit} & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & \mathbf{e}^T\boldsymbol{\ell}^{Credit} \end{bmatrix}
\begin{bmatrix} \mathbf{p} \\ w \end{bmatrix}
$$

$$
\equiv
\begin{bmatrix} \mathbf{A}^{Barter} & \boldsymbol{\ell}^{Barter} \\ \mathbf{0}_{1\times n} & 0 \end{bmatrix}
\begin{bmatrix} \mathbf{p} \\ w \end{bmatrix}
+ \mathbf{C}^{Barter}\mathbf{p} +
\begin{bmatrix} \mathbf{A}^{Debt}\mathbf{p} & \boldsymbol{\ell}^{Debt} w \\ \mathbf{0}_{1\times n} & 0 \end{bmatrix}
\begin{bmatrix} \mathbf{p} \\ w \end{bmatrix}
+ \mathbf{C}^{Debt}\mathbf{p}
\tag{3.8}
$$

The left and right hand sides of the above equation captures the revenues and expenditures, respectively, for the whole system.[13] The value of IOUs issued, i.e., the size of lending, that would allow the system to reach self-replacing is given by:

$$
\mathbf{\Delta Credit} = \begin{bmatrix} \mathbf{B}^{Credit} & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & \mathbf{e}^T\boldsymbol{\ell}^{Credit} \end{bmatrix}
\begin{bmatrix} \mathbf{p} \\ w \end{bmatrix} \tag{3.9}
$$

Equivalently, the size of borrowing is given by:

$$
\mathbf{\Delta Debt} = \underbrace{\begin{bmatrix} \mathbf{A}^{Debt} & \boldsymbol{\ell}^{Debt} \\ \mathbf{0}_{1\times n} & 0 \end{bmatrix}
\begin{bmatrix} \mathbf{p} \\ w \end{bmatrix}}_{\textit{Production financed with borrowing}}
+ \overbrace{\mathbf{C}^{Debt}\mathbf{p}}^{\textit{Consumption with borrowing}} \tag{3.10}
$$

We can express the accounting constraint associated with the general set of exchanges that allow for self-replacing as below:

[13] Note that the Barter values are uniquely determined by the prices and the wage rate. In the absence of deferred means of payments, the system could shrink to a lower level of production. Here we consider the prices, wage rate and IOUs that would allow exchanges to be such that the self-replacing condition is potentially fulfilled. This means that the total amount of the means of production bought using IOUs is uniquely determined.

$$
\overbrace{\overbrace{\begin{bmatrix} \mathbf{B}^{Barter} & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & \mathbf{e}^T\boldsymbol{\ell}^{Barter} \end{bmatrix}\begin{bmatrix} \mathbf{p} \\ w \end{bmatrix}}^{\textit{Sold with barter (value)}} + \overbrace{\mathbf{\Delta Credit}}^{\textit{Sold with IOUs}}}^{\textit{Total quantity sold (value)}}
$$

$$
= \underbrace{\underbrace{\begin{bmatrix} \mathbf{A}^{Barter} & \boldsymbol{\ell}_{n\times 1}^{Barter} \\ \mathbf{0}_{1\times n} & 0 \end{bmatrix}\begin{bmatrix} \mathbf{p} \\ w \end{bmatrix} + \mathbf{C}^{Barter}_{(n+1)\times n}\mathbf{p}}_{\textit{Bought with barter (value)}} + \underbrace{\mathbf{\Delta Debt}}_{\textit{Bought with IOUs}}}_{\textit{Total quantity bought (value)}} \tag{3.11}
$$

During the market day, in order to achieve a self-replacing state, producers might have to resort to *barter* exchanges and/or *credit* exchanges. In this set up, the following must always hold for each period:

i) $\mathbf{B} = \mathbf{B}^{Barter} + \mathbf{B}^{Credit}$: the total quantity produced must be sold either through barter exchanges or deferred means of payments, i.e., lending or credit.

ii) $\mathbf{A} = \mathbf{A}^{Barter} + \mathbf{A}^{Debt}$: the means of production must be bought either through *barter* exchanges or deferred means of payments, i.e., borrowing or debt.

iii) $\boldsymbol{\ell} = \boldsymbol{\ell}^{Barter} + \boldsymbol{\ell}^{Credit}$: labour is sold through *barter* or credit exchanges.

iv) $\mathbf{C} = \mathbf{C}^{Barter} + \mathbf{C}^{Debt}$: the produced surplus $\mathbf{s}$ is sold (or bought) either with *barter* or credit exchanges.[14]

v) $\mathbf{e}^T(\mathbf{\Delta Credit} - \mathbf{\Delta Debt}) = 0$: even when the vectors $\mathbf{\Delta Credit}$ and $\mathbf{\Delta Debt}$ are not $\mathbf{0}$, the sum of their differences must always be equal to zero.

3.4 Self-replacing prices with credit and debt

We move to the core of the paper to derive the self-replacing prices in the presence of credit and debt. Each producer i can buy the consumption vector $\mathbf{c}_i$ if they have the purchasing power to do so. The industries and workers would have this purchasing power if the prices, wage rate, lending and borrowing are such that:

$$
\mathbf{d}\mathbf{s}^T\mathbf{p} = \mathbf{d}\mathbf{e}^{\mathbf{T}}\mathbf{C}\mathbf{p} = \begin{bmatrix} d_1\mathbf{s}^T\mathbf{p} \\ d_2\mathbf{s}^T\mathbf{p} \\ \vdots \\ d_n\mathbf{s}^T\mathbf{p} \\ d_W\mathbf{s}^T\mathbf{p} \end{bmatrix} = \overbrace{\begin{bmatrix} b_1p_1 - \mathbf{a}_1\mathbf{p} - \ell_1 w - (\Delta Credit_1 - \Delta Debt_1) \\ b_2p_2 - \mathbf{a}_2\mathbf{p} - \ell_2 w - (\Delta Credit_2 - \Delta Debt_2) \\ \vdots \\ b_np_n - \mathbf{a}_n\mathbf{p} - \ell_n w - (\Delta Credit_n - \Delta Debt_n) \\ \mathbf{e}^T\boldsymbol{\ell}w - (\Delta Credit_W - \Delta Debt_W) \end{bmatrix}}^{\textit{Purchasing Capacity}} \tag{3.12}
$$

Written in a more compact form, we have:

[14] The matrix $\mathbf{C}$ has $(n+1)$ rows and n columns. Each row is the consumption actual demand $\mathbf{c}_z$ of the producers ($z = 1, \dots, n$) or the workers ($z = n+1$). Given the self-replacing condition and the fact that net investment is zero by assumption, we have that the sum by columns of the matrix $\mathbf{C}$ must be equal to the surplus $\mathbf{s}$, i.e., $\mathbf{s} = \mathbf{C}^T\mathbf{e}$.

$$\mathbf{ds}^T\mathbf{p} = \mathbf{de^T C p} = \begin{bmatrix} (\mathbf{B}-\mathbf{A})\mathbf{p} - \boldsymbol{\ell} w \\ \mathbf{e}^T \boldsymbol{\ell} w \end{bmatrix} - (\Delta\mathbf{Credit} - \Delta\mathbf{Debt}) \tag{3.13}$$

If the prices and the wage rate are expressed in terms of the surplus being set as the *numéraire*, we have that $\mathbf{s}^T\mathbf{p}$=1 (as in sec.2.3):

$$\mathbf{d} = \overbrace{\begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \\ d_W \end{bmatrix}}^{\substack{\textit{Distribution} \\ \text{(Physical and value)}}} = \overbrace{\underbrace{\left[\begin{array}{c|c} (\mathbf{B}-\mathbf{A}) & -\boldsymbol{\ell} \\ \hline \mathbf{0}_{1\times n} & \mathbf{e}^T\boldsymbol{\ell} \end{array}\right] \left[\begin{array}{c} \mathbf{p} \\ \hline w \end{array}\right]}_{\textit{Income}} - \underbrace{\begin{bmatrix} \boldsymbol{\Delta}\mathbf{Credit} - \boldsymbol{\Delta}\mathbf{Debt} \end{bmatrix}}_{\textit{Lending and Borrowing}}}^{\textit{Purchasing Capacity}} \tag{3.14}$$

3.5 Computation of prices, wage rates and profit rates

The domain of all possible vectors $[\mathbf{p}, w]^T$ for which the system is in a self-replacing state can be computed by considering all possible distributions of $\mathbf{d}$ in combination with all possible feasible changes in the credit-debt conditions ($\boldsymbol{\Delta}\mathbf{Credit} - \boldsymbol{\Delta}\mathbf{Debt}$). The knowledge of the distribution vector, lending and borrowing among the industries and workers, together constitute enough information for the computation of prices, wage rates and profit rates that would allow self-replacing.

$$\begin{bmatrix} \mathbf{p} \\ w \end{bmatrix} = \begin{bmatrix} (\mathbf{B}-\mathbf{A}) & -\boldsymbol{\ell} \\ \mathbf{0}_{1\times n} & \mathbf{e}^T\boldsymbol{\ell} \end{bmatrix}^{-1} \big(\mathbf{d} + \boldsymbol{\Delta}\mathbf{Credit} - \boldsymbol{\Delta}\mathbf{Debt}\big) \tag{3.15}$$

We have derived the expression to calculate the prices and wage rate that permit the system to be self-replacing. Once the inverted matrix is expanded, we have the following:

$$\begin{bmatrix} \mathbf{p} \\ w \end{bmatrix} = \begin{bmatrix} (\mathbf{B}-\mathbf{A})^{-1} & -\dfrac{\boldsymbol{\ell}}{\mathbf{e}^T\boldsymbol{\ell}} \\ \mathbf{0}_{1\times n} & \dfrac{1}{\mathbf{e}^T\boldsymbol{\ell}} \end{bmatrix} \big(\mathbf{d} + \boldsymbol{\Delta}\mathbf{Credit} - \boldsymbol{\Delta}\mathbf{Debt}\big)$$

$$= \begin{bmatrix} (\mathbf{RA})^{-1} & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & \dfrac{1}{\mathbf{e}^T\boldsymbol{\ell}} \end{bmatrix} \big(\mathbf{d} + \boldsymbol{\Delta}\mathbf{Credit} - \boldsymbol{\Delta}\mathbf{Debt}\big) \tag{3.16}$$

Eq.(3.16) is at the core of our analysis. Contrast this with eq.(2.14), which corresponds to the case in which there is no credit and debt. This can be written explicitly in terms of the distribution and financial balances of the industries and workers separately:

$$\mathbf{p} = (\mathbf{RA})^{-1}\big(\mathbf{d}_{n\times 1} + \boldsymbol{\Delta}\mathbf{Credit}_{n\times 1} - \boldsymbol{\Delta}\mathbf{Debt}_{n\times 1}\big) \tag{3.17}$$

$$w = \frac{1}{\mathbf{e}^T\boldsymbol{\ell}}\big(d_W + \Delta\text{Credit(W)} - \Delta\text{Debt(W)}\big) \tag{3.18}$$

3.5.1 Determination of the domains of d, ΔCredit and ΔDebt

The consumption in the system (by both owners of industries and workers) is summarised by matrix $\mathbf{C}$, where z is the index referring to both owners of industries (i) and workers (W).

$$\mathbf{C} = \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \vdots \\ \mathbf{c}_n \\ \mathbf{c}_W \end{bmatrix} = \begin{bmatrix} c_1^1 & c_1^2 & \dots & c_1^j & \dots & c_1^n \\ c_2^1 & c_2^2 & \dots & c_2^j & \dots & c_2^n \\ \vdots & & & & & \\ c_n^1 & c_n^2 & \dots & c_n^j & \dots & c_n^n \\ c_W^1 & c_W^2 & \dots & c_W^j & \dots & c_W^n \end{bmatrix} \tag{3.19}$$

We know that the social consumption is out of the total surplus of the system, i.e., $\mathbf{s} = \mathbf{C^T e}$. The distributional vector $\mathbf{d}$, as in eq.(2.13), denotes the share of the value of the produced surplus $\mathbf{s}$ that goes to owners of industries and workers. It is possible to interpret the elements d_z *as if* they were equivalent to fractions of the physical surplus $\mathbf{s}$ that is actually consumed by the producers and the workers.

The rationale is the following: the amount spent by each z in consumption is $\mathbf{c}_z\mathbf{p}$ and given that there is no investment, we have $d_z\mathbf{s}^T\mathbf{p} = \mathbf{c}_z\mathbf{p}$. Taking advantage of fact that the condition $\mathbf{s} = \mathbf{C^T e}$ must hold, we can treat $d_z\mathbf{s}^T = d_z\mathbf{c}_z$ for our purpose of determining the *total domain* of self-replacing prices. This is because when $d_z\mathbf{s} \neq \mathbf{c}_z$, there will be an imbalance, and for $\mathbf{s} = \mathbf{C^T e}$ to hold, there need to be an exact compensation elsewhere. In other words, there must be other producers or workers who match this difference $(\mathbf{c}_z - d_z\mathbf{s})$ exactly (but with the opposite sign).

Therefore, for analytical clarity and simplification of the exposition, we can use the expressions in value and physical terms interchangeably, as if they are equivalent. This equivalence does not modify the computations of the self-replacing prices, wage rates and profit rates and financial balances. In the remainder of the paper, we treat each element d_z of the vector $\mathbf{d}$ *as if* it were a fraction of the physical surplus $\mathbf{s}$ and not just a fraction of its value.

The variations in the amount of existing credit and debts contracts (see sec. 3.3.1) can happen only against an exchange with commodities in our set up. If the desired commodities are not available, then agents cannot buy or sell them, consequently, no new lending or borrowing can take place.

The boundary conditions for the vector $\mathbf{d} + \mathbf{\Delta Credit} - \mathbf{\Delta Debt}$ in eqn.(3.16) are the following:

(i) each element of the distribution vector is greater than or equal to zero

$$d_z \geq 0 \quad \forall z \in (1, n+1).$$

(ii) the sum of all the elements in the distribution vector must be equal to 1,

$$\sum_{z=1}^{n+1} d_z = 1;$$

(iii) the sum of all the elements in the credit-debt vector is equal to 0,

$$\sum_{z=1}^{n+1} (\Delta Credit_z - \Delta Debt_z) = 0$$

(iv) the maximum possible increase in credit (or debt) for the whole system is the value of the surplus vector, i.e., 1.

$$\sum_{z=1}^{n+1} \Delta Credit_z = 1 \ \ (or \sum_{z=1}^{n+1} \Delta Debt_z = 1)$$

(v) each element of the credit-debt vector lies between -1 and 1,

$$-1 \leq (\Delta Credit_z - \Delta Debt_z) \leq 1$$

Conditions (i)-(ii) are satisfied by definition. Condition (iii) follows from the accounting the accounting rules and the nature of IOUs. Conditions (iv)-(v) are a consequence of the particular choice of the *numéraire*, the self-replacing assumption and the (physical) limits on borrowing and lending.

The cardinality of the set of all possible distribution vectors is determined by the number of commodities n, the chosen grid size h of the interval, $d_z \in [0, 1]$. The number of combinations possible increase as the grid size h gets smaller.[15] Although the number of distributional vectors grows exponentially as the grid size decreases, the set of all possible distributions is always finite, i.e. enumerable.[16] The conditions (iii)-(v) above also imply that the number of the vectors in $\mathbf{\Delta Credit} - \mathbf{\Delta Debt}$ is finite and the cardinality of the set is of the same order as that of the distribution vectors.

The set of all the combinations of $\mathbf{d}$ and $\mathbf{\Delta Credit} - \mathbf{\Delta Debt}$ associated with self-replacing is independent of prices, wage and profit rates, and this can be computed *a priori* under conditions (i)-(v). Note that we are referring to the entire set and not the individual elements as being independent of prices. We have now provided a constructive procedure to compute the set of all self-replacing prices for the system, using eqn. (3.16).

3.6 Deferred payments and time sequences

So far, we have examined the determination of self-replacing prices at a given period with deferred means of payments. However, deferred means of payments are associated with a future time period in which these are written off and eventually new contracts may emerge. IOUs may be generated and transferred from one period to another and this might influence the set of self-replacing prices in the subsequent periods. Equally, the existing prices can influence the set of feasible credit and debt vectors associated with self replacing. Therefore, it becomes necessary to understand the effects of pre-existing future promises to pay on the determination of the set of self-replacing prices.

The evolution of the IOUs would be the following:

$$\begin{aligned}
\mathbf{F}_t^A &= \mathbf{\Delta Credit}_t + (1 + i_t^F)\mathbf{F}_{t-1}^A \\
\mathbf{F}_t^L &= \mathbf{\Delta Debt}_t + (1 + i_t^F)\mathbf{F}_{t-1}^L \\
\mathbf{F}_t^B &= \mathbf{F}_t^A - \mathbf{F}_t^L \\
&= \mathbf{\Delta Credit}_t - \mathbf{\Delta Debt}_t + (1 + i_t^F)(\mathbf{F}_{t-1}^A - \mathbf{F}_{t-1}^L)
\end{aligned} \tag{3.20}$$

where: $\mathbf{F}_t^A, \mathbf{F}_t^L, \mathbf{F}_t^B$ are each $(n+1) \times 1$ vectors representing assets, liabilities and balances,

[15] For a chosen grid size h, the elements of distribution vector is given by $d_z \in \{0, h, 2h, \ldots, 1 - h, 1\}$.

[16] For example, for three commodities and $h = 0.1$, the number of distinct distribution vectors is 275. For $h = 0.01$, this increases to 173,340, and for $h = 0.001$ this becomes 167,605,051, and so on.

respectively, of the stock of deferred means of payments at the end of a given accounting period t (or beginning of period $t+1$); $\Delta\mathbf{Credit}_t$ and $\Delta\mathbf{Debt}_t$ represent changes in credit and debt as defined in eqs.(3.9) and (3.10), respectively; i_t^F is an exogenous interest rate on financial contracts.[17]

We reiterate that both $\Delta\mathbf{Credit}_t$ and $\Delta\mathbf{Debt}_t$ are conceived as exchanges of real or physical commodities against financial means of payment, as described above in eqs. (3.9) and (3.10) respectively. The total variations of the credit and debt position of the industries (or workers) would also depend on the interest rate that prevails in the financial contracts. Therefore, we have to consider the changes in the financial positions due to revenues accrued and expenditures made in the form of IOUs. These are summarised as follows:

$$\begin{aligned}
\Delta\mathbf{F}_t^A &= \mathbf{F}_t^A - \mathbf{F}_{t-1}^A = \Delta\mathbf{Credit}_t + i_t^F\mathbf{F}_{t-1}^A \\
\Delta\mathbf{F}_t^L &= \mathbf{F}_t^L - \mathbf{F}_{t-1}^L = \Delta\mathbf{Debt}_t + i_t^F\mathbf{F}_{t-1}^L \\
\Delta\mathbf{F}_t^B &= \mathbf{F}_t^B - \mathbf{F}_{t-1}^B
\end{aligned} \tag{3.21}$$

Here $\Delta\mathbf{F}_t^A$ and $\Delta\mathbf{F}_t^L$ are the total variations of positive and negative financial balances, respectively. The time evolution of debt and credit can be represented in the following way:

[17] It is possible to have heterogeneous interest rates that vary by sectors or industries involved. In that case i^F would have to be a vector. For the simplicity of the exposition, we assume that the the monetary or financial interest rate i_t^F is uniform.

$\boxed{\textbf{Period 1}}$

$$\xrightarrow{production} \overbrace{\mathbf{A}_0, \boldsymbol{\ell}_0, \mathbf{F}_0^B, \quad \overbrace{\mathbf{b}_1}^{\textit{first harvest}}}^{\textit{production cycle=1}} \xrightarrow{exchange}$$

$$\xrightarrow{exchange} \{\mathbf{p}_1, w_1, i_1^F\} \rightarrow \overbrace{\mathbf{A}_1, \boldsymbol{\ell}_1, \mathbf{s}_1, \mathbf{d}_1, \mathbf{C}_1, \Delta\mathbf{F}_1^B}^{\textit{first market day}} \xrightarrow{production}$$

$\boxed{\textbf{Period 2}}$

$$\xrightarrow{production} \overbrace{\mathbf{A}_1, \boldsymbol{\ell}_1, \mathbf{F}_1^B, \quad \overbrace{\mathbf{b}_2}^{\textit{harvest}}}^{\textit{production cycle=2}} \xrightarrow{exchange}$$

$$\xrightarrow{exchange} \{\mathbf{p}_2, w_2, i_2^F\} \rightarrow \overbrace{\mathbf{A}_2, \boldsymbol{\ell}_2, \mathbf{s}_2, \mathbf{d}_2, \mathbf{C}_2, \boldsymbol{\Delta}\mathbf{F}_2^B}^{\textit{second market day}} \xrightarrow{production} \tag{3.22}$$

$$\rightarrow \cdots \rightarrow \cdots \rightarrow \cdots \rightarrow \cdots \rightarrow \cdots \rightarrow \cdots \rightarrow \cdots \rightarrow$$

$\boxed{\textbf{Period } T}$

$$\xrightarrow{production} \overbrace{\mathbf{A}_{T-1}, \boldsymbol{\ell}_{T-1}, \mathbf{F}_{T-1}^B \quad \overbrace{\mathbf{b}_T}^{\textit{harvest}}}^{\textit{production cycle=T}} \xrightarrow{exchange}$$

$$\xrightarrow{exchange} \{\mathbf{p}_T, w_T, i_T^F\} \rightarrow \overbrace{\mathbf{A}_T, \boldsymbol{\ell}_T, \mathbf{s}_T, \mathbf{d}_T, \mathbf{C}_T, \boldsymbol{\Delta}\mathbf{F}_T^B}^{\textit{market day (T)}}$$

The above representation shows the initial conditions $\mathbf{A}_0, \boldsymbol{\ell}_0, \mathbf{F}_0^B, \mathbf{b}_1$ of the economic system and the sequences of triplets (i.e., prices, wage rate, and interest rate) for each period t, where $t \in [0, 1, \ldots T]$ and T is the final time period under consideration:

$$\{(\mathbf{p}_1, w_1, i_1^F), (\mathbf{p}_2, w_2, i_2^F), \ldots (\mathbf{p}_T, w_T, i_T^F)\}$$

As it stands, eq.(3.22) is not necessarily restricted to being in the self-replacing condition. We have kept this more general notation to emphasise the generality of our approach. The self-replacing condition is a special case of the above sequence, wherein:

$$\begin{array}{ccccccccccccc}
\mathbf{b} & = & \mathbf{b}_0 & = & \mathbf{b}_1 & = & \mathbf{b}_2 & = & \ldots & = & \mathbf{b}_T & ; \\
\mathbf{A} & = & \mathbf{A}_0 & = & \mathbf{A}_1 & = & \mathbf{A}_2 & = & \ldots & = & \mathbf{A}_T & ; \\
\boldsymbol{\ell} & = & \boldsymbol{\ell}_0 & = & \boldsymbol{\ell}_1 & = & \boldsymbol{\ell}_2 & = & \ldots & = & \boldsymbol{\ell}_T & ; \\
\mathbf{s} & = & \mathbf{s}_0 & = & \mathbf{s}_1 & = & \mathbf{s}_2 & = & \ldots & = & \mathbf{s}_T & ;
\end{array} \tag{3.23}$$

In this special case, the composition of surplus $\mathbf{s}$ is constant. However, this does not in any way imply that the distribution of the surplus, $\mathbf{d}_t$ is constant. In fact, it would depend on (a) the prices and wage rate relative to the period considered; (b) existing and newly generated deferred means of payments, which in turn will also depend on the interest rate on financial contracts, i^F (see eq.3.20).

As in Sraffa's PCMC here we do not provide a theory of prices or distribution. In the same spirit, neither do we provide a theory concerning the determination of specific credit-debt structure and interest rates.[18] Instead, for given distribution of the surplus, past methods of production, available labour and output, our aim is to determine the set of prices and the possible financial conditions that would allow the system to reproduce itself. When deferred means of payments and the rules concerning the exact form of repayments are given, we conclude that the set of self-replacing prices is in fact path-dependent. Hence, it becomes evident that the availability of credit and debt play an important role in determining the viability of the system and by having an effect on distribution.

4 Self-replacing prices: numerical example

In this section, we demonstrate the above ideas through a numerical example. We use the same numbers from Sraffa's PCMC (p.19) to identify self-replacing prices in the presence of credit and debt.

If we exclude borrowing and lending, there are two distinct possibilities to explore: (1) the case with uniform rate of profits assumption (as in the traditional Sraffian Schemes in PCMC); (2) the case with non-uniform rates of profit. The second case has been analysed in Zambelli (2018b). In these cases, all exchanges are solely barter as defined in eq.3.3 and distribution is computed as in eq.(2.13).[19]

In order to identify a general set of self-replacing prices and wage rates, with and without deferred payments, we start with an arbitrary vector. Once a specific vector of prices and wage rates is given, say $\{\bar{\mathbf{p}}, \bar{w}\}$, there are two scenarios that would allow self-replacing (see eqs. 3.12, 3.13).

Case 1: the distribution (i.e., consumption of the surplus by the actors) will be such that the purchasing power of the goods sold by the owners of the means of production is exactly equal to the values of the quantities bought. In terms of equation 3.4, this is case where revenues are entirely constituted by barter revenues and the credit revenues are 0, i.e. *Revenues*$(\cdot)$ = *Barter Revenues*(.). By symmetry, *Expenditures*$(\cdot)$ = *Barter Expenditures*$(\cdot)$. In this case, the credit and debt positions do not change due to lack of purchasing power of the owners of the means of production, $\mathbf{\Delta Credit} - \mathbf{\Delta Debt} = \mathbf{0}$.[20]

Case 2: the prices and wage rate $\{\bar{\mathbf{p}}, \bar{w}\}$ are such that the revenues from the selling of the means of production (including labour) is not matched exactly with the values of the effective demands. In this case, deferred means of payments, i.e., credit and debt, are necessary to compensate the mismatch. In the following accounting periods, these debts need to be paid back (and credits cashed in), along with the sale and purchase of commodities. The changes in purchasing power created by repayment of debt would have to be accompanied by other changes that would keep the system in self-replacing condition. This can happen either through changes in distribution or through changes in prices. In other words, it is achieved either with a reduction in consumption by those that have to pay back the debt (i.e., a change in distribution $\mathbf{d}$) or, alternatively, if there

[18] These are susceptible to being determined by forces outside the system of production. For a discussion, see Venkatachalam and Zambelli (2021, sec.3).

[19] Alternatively, distribution can be calculated using eq.(3.14) for the case in which there are no deferred payment contracts, $\mathbf{\Delta Credit} = \mathbf{0}$ and $\mathbf{\Delta Debt} = \mathbf{0}$.

[20] If previously issued deferred means of payments exists and the interest rate on financial contracts i_t^F is different from zero, then the evolution of credit and debt positions follow the dynamics in eqs. 3.20 and 3.21. When the expenditures are exclusively barter, the changes in financial positions happen exclusively due to financial interest rate and debt servicing.

is a new set of prices $\{\bar{\bar{\mathbf{p}}}, \bar{\bar{w}}\}$ that allow the producers and the workers to pay back the debt and maintain the previous level of consumption.

Let us consider these possibilities of self-replacing using an example in PCMC, p. 19.

$$\mathbf{A} = \begin{bmatrix} 90 & 120 & 60 \\ 50 & 125 & 150 \\ 40 & 40 & 200 \end{bmatrix}; \boldsymbol{\ell} = \begin{bmatrix} \frac{3}{16} \\ \frac{5}{16} \\ \frac{8}{16} \end{bmatrix}; \mathbf{b} = \begin{bmatrix} 180 \\ 450 \\ 480 \end{bmatrix}; \tag{4.1}$$

The rows of $\mathbf{A}$ represent iron, coal and wheat industries, respectively. The columns of $\mathbf{A}$ represent the means of production used as inputs by the industries. At the end of the production period, the quantities $\mathbf{b} = [180, 450, 480]^T$ have to be exchanged to organize production for the subsequent period. The outcomes of the market day are not known apriori, but if production structure of the previous year were to be replicated, the gross production $\mathbf{b}_i$ has to be exchanged in such a way that producers i can buy the means of production $\mathbf{a}_i$ and labour ℓ_i. In this specific example, at the end of the market day, the surplus to be distributed would have to be $\mathbf{s} = [0, 165, 70]^T$. This is derived as the difference between the gross output $\mathbf{b}$ and the means of production used to produce it.

4.1 Conditions for self-replacement

We will now determine the prices, wage rates, profit rates and credit-debt positions that would allow self-replacing. Note that credit-debt relations in our framework refer to the existence of obligations that link different points in time.

As shown earlier (see eq. 3.15), by choosing the surplus as the *numéraire* and given any triple $\mathbf{A}, \boldsymbol{\ell}, \mathbf{b}$, the prices and wage rates that would allow the system to be in a self-replacing state can be computed provided the following conditions hold:

(i) each element of the distribution vector is greater than or equal to zero;

$$d_z \geq 0 \quad \forall z \in (1, n+1)$$

(ii) sum of all the elements in the distribution vector is equal to 1; $\sum_{z=1}^{n+1} d_z = 1$;

(iii) the vector $\mathbf{\Delta Credit} - \mathbf{\Delta Debt}$ is such that it provides the necessary deferred means of payments to allow the exchanges when barter exchanges are not self-replacing.[21]

The first two conditions are satisfied by construction and by setting the net surplus as the *numéraire*. From the computational point of view, for any given distribution $\mathbf{d}$ of surplus $\mathbf{s}$, we need to identify the cloud of price vectors, $\mathbf{p_d}$, wage rate $w_\mathbf{d}$, profit rates and variations in $\mathbf{\Delta Credit} - \mathbf{\Delta Debt}$ as per eq. 3.16. These are shown in figs. 4.1 and 4.2.

4.2 Domain of prices and wage-rate

Figure 4.1 indicates the pairwise domain of prices (and wage rate) of different goods which are associated with all possible, feasible self-replacing distribution vectors.The sub-figures (a)-(d) in fig. 4.1 indicate, respectively, the domains for: (a) the prices of coal and iron; (b) wage rate (or share to workers) and price of iron; (c) prices of coal and wheat; (d) wage rate and price of wheat. When individual prices fall outside the domain, it implies

[21] The case in which $\mathbf{\Delta Credit} - \mathbf{\Delta Debt} = \mathbf{0}_{(n+1)\times 1}$ is examined in Zambelli (2018b).

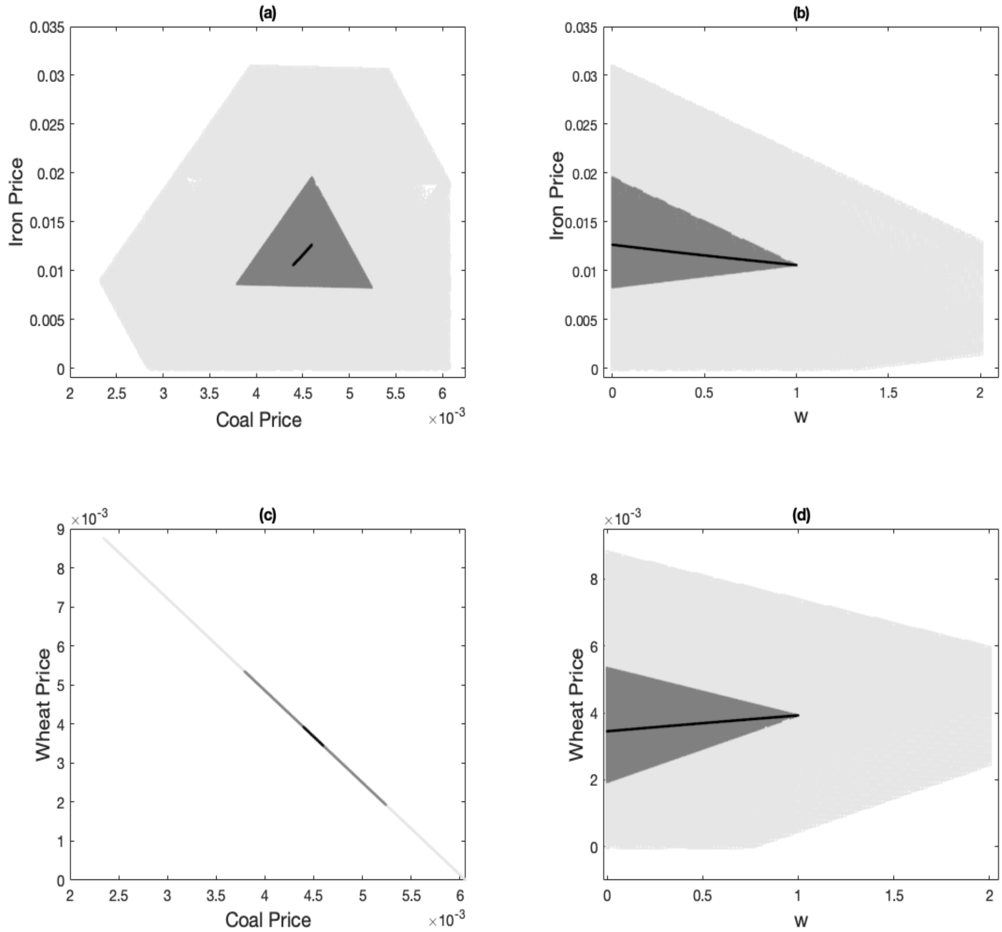

Figure 4.1: ***Domain of the self-replacing prices and wage rate:*** *Subfigures denote the domain of the prices of commodities and the wage rate associated with all possible feasible self-replacing distribution vectors. The light grey, dark grey and black regions denote, respectively, the set of prices that allow the system to be in (i) self-replacing state, (ii) self-replacing state without the emergence of credit and debt and (iii) subset of the self-replacing prices associated with a uniform rate of profits. w denotes the wage rate (or share to workers of the surplus). The triple $\mathbf{A}$, $\boldsymbol{\ell}$, $\mathbf{b}$ used for the computations of these regions are based on numbers in 4.1 (or PCMC, p. 19).* Numéraire*: surplus vector, $\mathbf{s} = [0\ t.iron,\ 165\ t.coal,\ 70\ qr.wheat]^T$. The elements of prices and wage rate assuring self-replacement are not independent. They are determined by eq. 3.15.*

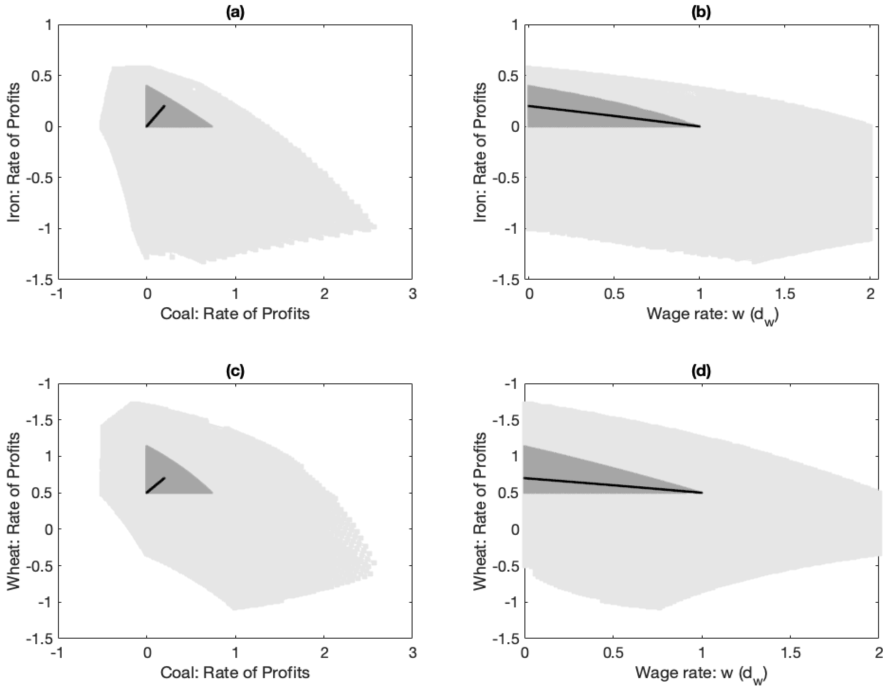

Figure 4.2: ***Domain: rates of profits and wage rates:*** *Subfigures denote the domain of rates of profits and wage rate associated with all possible feasible self-replacing distribution vectors. The light grey, dark grey and black regions denote, respectively, the set of prices that allow the system to be in (i) self-replacing state, (ii) self-replacing state without the emergence of credit and debt and (iii) subset of the self-replacing prices associated with a uniform rate of profits. w denotes the wage rate (or share to workers of the surplus). The triple* $\mathbf{A}$, $\boldsymbol{\ell}$, $\mathbf{b}$ *used for the computations of these regions are based on numbers in 4.1 (or PCMC, p. 19).* Numéraire*: surplus vector,* $\mathbf{s} = [0\ t.iron,\ 165\ t.coal,\ 70\ qr.wheat]^T$. *The elements of the quadruple of rates of profits and wage rate assuring self-replacing are not independent. Their rates of profits are computed with eq. 3.18 and the wage rate with eq. 3.18.*

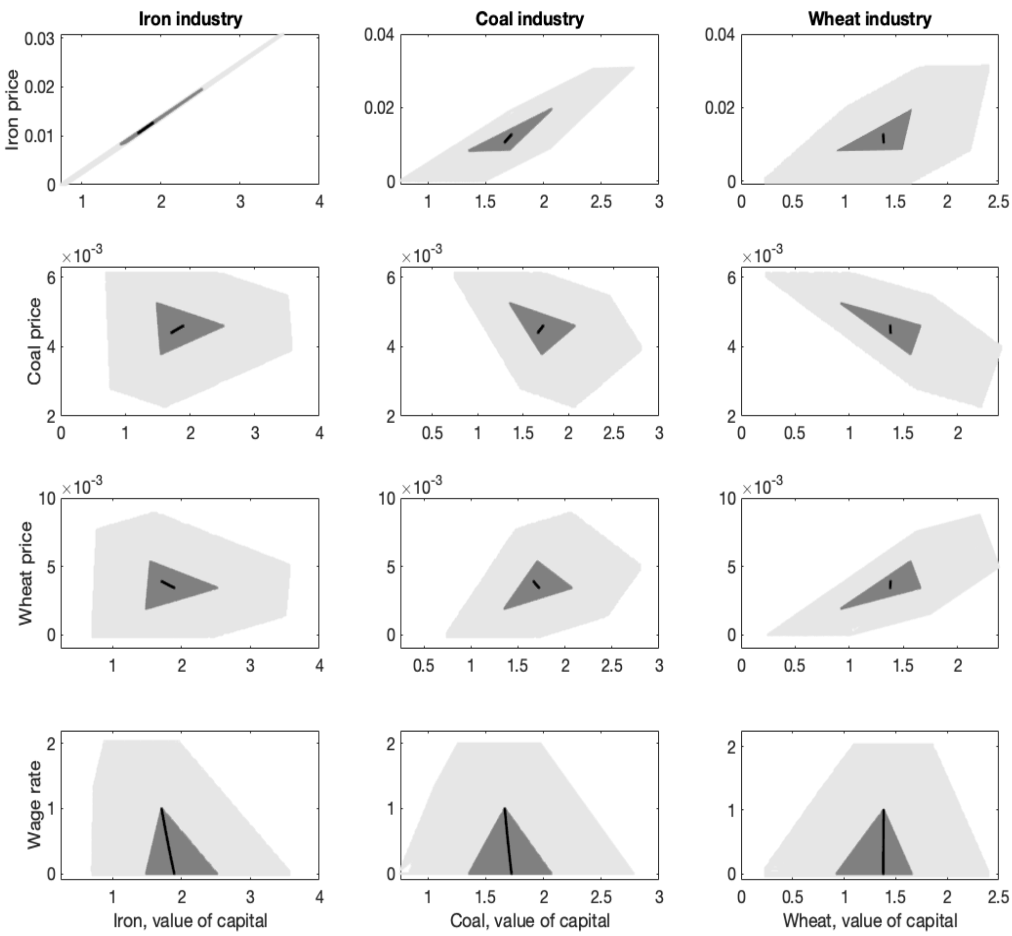

Figure 4.3: ***Domain: Value of capital (industry), self-replacing prices and wage rate.*** *Sub-figures denote the value of capital for individual industries, i.e. the value of the output produced divided by the value of the capital ratios, in relation with self-replacing prices (see 4.2). The values associated with individual industries (iron, coal and wheat) are organised by columns. The light grey, dark grey and black regions denote, respectively, the set of prices that allow the system to be in (i) self-replacing state, (ii) self-replacing state without the emergence of credit and debt and (iii) subset of the self-replacing prices associated with a uniform rate of profits. w denotes the wage rate (or share to workers of the surplus). The triple* $\mathbf{A}$, $\boldsymbol{\ell}$, $\mathbf{b}$ *used for the computations of these regions are based on numbers in 4.1 (or PCMC, p. 19).* Numéraire*: surplus vector,* $\mathbf{s} = [0 \ t.iron,\ 165 \ t.coal,\ 70 \ qr.wheat]^T$*. Note that once the value of capital for an industry is picked, the subset of self-replacing prices is determined. Alternatively, once a vector of prices and wage rate is given, as in Figure 4.1, the value of capital per industry is determined.*

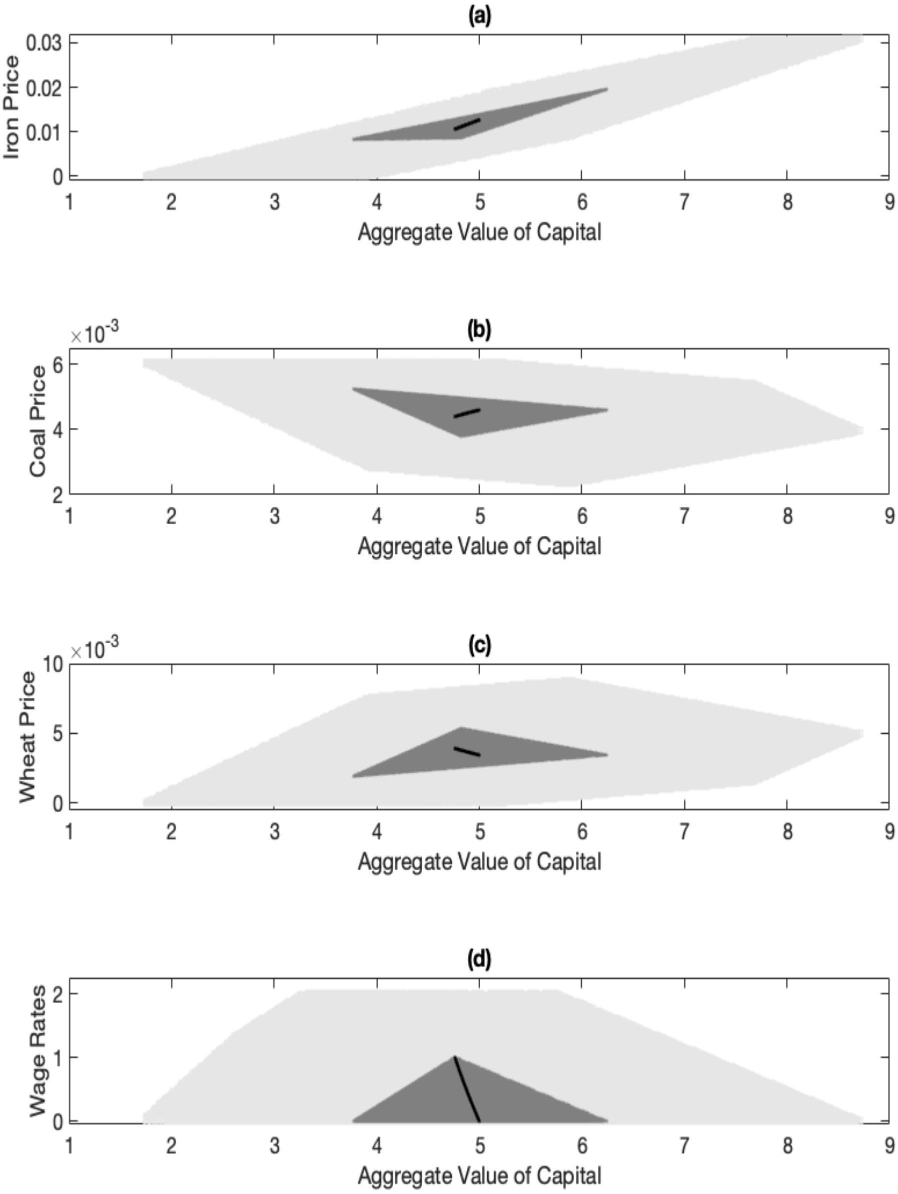

Figure 4.4: ***Domain: aggregate value of capital, prices and wage rate.*** *The subfigures denote the domain of aggregate value of capital (the sum of the values of the physical means of production over all industries), price of different commodities and the wage rate. The light grey, dark grey and black regions denote, respectively, the set of prices that allow the system to be in (i) self-replacing state, (ii) self-replacing state without the emergence of credit and debt and (iii) subset of the self-replacing prices associated with a uniform rate of profits. w denotes the wage rate (or share to workers of the surplus). The triple* $\mathbf{A}$, $\boldsymbol{\ell}$, $\mathbf{b}$ *used for the computations of the triangles was given in 4.1 (or PCMC, p. 19).*Numéraire*: surplus vector,* $\mathbf{s} = [0 \text{ t.iron}, 165 \text{ t.coal}, 70 \text{ qr.wheat}]^T$. *Once the value of capital is picked, the subset of self-replacing prices is determined. Alternatively, once a vector of prices and wage rate is fixed as in Figure 4.1, the value of capital is determined. The value of the aggregate output would be the value of capital plus the value of the physical net product or surplus (which is the* numéraire *and therefore is by definition equal to 1).*

that the economic system cannot self-replicate under the required conditions. The inner black region denotes the subset of the self-replacing prices associated with a uniform rate of profits. The dark grey region denotes the set of self-replacing prices under non-uniform rates of profit. In this region, however, there is no emergence of credit and debt. The larger, outer light grey area denotes the set of all self-replacing prices, including those that would require the emergence of new credit-debt relations. The triple $\{\mathbf{A}, \mathbf{b}, \boldsymbol{\ell}\}$ used for the computations of these regions was given in 4.1 (also in PCMC, p. 19). Given our choice of the *numéraire*, the value of the wage rate (w) also indicates the share of the surplus going to the labourers (d_W). It is important to note that the prices and wage rate associated with self-replacement are not completely independent. They are determined simultaneously with the distribution (see eq.3.15).

4.3 Domain of profit and wage rates

Figure 4.2 indicates the domain of the rates of profit. The sub-figures (a)-(d) in fig. 4.2 indicate the domains for: (a) rates of profits of coal and iron; (b) wage rate (or share to workers) and rates of profit of iron; (c) rates of profits of coal and wheat;(d) wage rate and rates of profit for wheat. As in Figure 4.1, for the system to be in a self-replacing condition, the rates of profits have to fall within the light grey area. The domain of prices associated with uniform rate of profits is identified by the black line. We note that this domain, related to uniform rate of profits assumption, is the case presented in PCMC. As we can see from the figure, this is a much smaller subset of all the possible combinations that would allow the system to self-replicate. The dark grey area identifies the cases in which there are non-uniform rates of profit, however, with no emergence of credit and debt. The outer grey area, as before, expands this domain by allowing for the possibility of having credit and debt as well. The elements of the quadruple of rates of profits and wage rate assuring self-replacing are not independent. The prices and wages are computed using eq. 3.17 and 3.18. As before, the surplus vector $\mathbf{s}$ is taken to be the *numéraire*.

4.4 Aggregate value of capital and output-capital ratios

The value of the aggregate capital used for the production of the same level output and surplus as in the previous production cycle is of interest. Though the physical quantities of the means of production do not change, the distribution may. Changes in distribution, in turn, affect the set of self-replacing prices as they are simultaneously determined. We show the results concerning the *aggregate* value of capital, i.e., the sum of the values of the physical means of production over all industries. More specifically, we show the relationship between aggregate value of capital and the self-replacing prices of different commodities (and wage rate). The three different coloured regions retain the same interpretations as in the previous sections concerning uniform rate of profits, non-uniform rates with and without the emergence of credit and debt. To each specification of aggregate value of capital, there is set of associated price vectors and distributions. Once we fix a value of aggregate capital, the corresponding price vectors can be identified in the associated domain. For each industry i, the value of capital k_i is determined as follows:

$$\mathbf{k} = \overbrace{\begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}}^{\substack{\textit{Value of capital} \\ \textit{(industry)}}} = \begin{bmatrix} \mathbf{a_1 p} \\ \mathbf{a_2 p} \\ \vdots \\ \mathbf{a_n p} \end{bmatrix} = \mathbf{Ap} \tag{4.2}$$

Aggregate value of capital for the whole system is given by:

$$k^{agg} = \mathbf{e}^T \mathbf{Ap} \tag{4.3}$$

Figures 4.3 and 4.4 are graphical illustrations of the important critique of neoclassical economic theory presented in PCMC. As we can see clearly from the figure 4.3, the value of capital for the industries would vary with different prices or distributions. Figure 4.4 captures this for the aggregate level of the economy. These figures, in our opinion, demonstrate Sraffa's critique in PCMC (p. 38) for a more general setting with credit and debt:[22]

> '. . . the movement of relative prices, in the face of unchanged methods of production, cannot be reconciled with any notion of capital as a measurable quantity independent of distribution and prices.'

This is not only a demonstration of a critique, but also a constructive result. It shows how to link values with distribution, where distribution is not computed as an index or in nominal terms, instead as a fraction of the physical surplus (or a bundle of commodities and services) available to the society. The dependence of the value of capital on the distribution should not be mistaken to be a result pertaining exclusively to the macro-level. It is equally applicable to the values of the means of production used by individual industries and the firms as shown in fig.4.3.

4.5 Debt extinction

Let us consider the case where exchanges have taken place utilising deferred means of payments. We take the values indicated in table 4.1 as a starting point (Period 0) of the analysis. In this example, we start with uniform rate of profits across industries at period 0. If we require the same distribution to be preserved in the subsequent period, i.e., period 1, we can examine whether the system can self-replicate for the same set of prices, wage rates and profit rates. This may not always be possible without credit and debt.[23] If the prices and the distribution take values as in table 4.2, issuance of deferred

[22]See Zambelli (2018a) for a theoretical discussion and an empirical demonstration of the impossibility of a measurement of aggregate capital that is independent of distribution and prices. This raises important questions about the applicability of notions such as aggregate production function. Note that the notion of aggregate production function is general because it can be extended to all the cases where capital is composite: firms, industries, and the whole economy.

[23]For the chosen distribution to be preserved for the given prices, the credit and debt positions are computed by rearranging eq.(3.14) as below:

$$\mathbf{\Delta Credit} - \mathbf{\Delta Debt} = \left[\begin{array}{c:c} (\mathbf{B} - \mathbf{A}) & -\boldsymbol{\ell} \\ \hdashline \mathbf{0}_{1\times n} & \mathbf{e}^T\boldsymbol{\ell} \end{array}\right] \left[\begin{array}{c} \mathbf{p} \\ \hdashline w \end{array}\right] - \mathbf{d}$$

Table 4.1: **Period 0 (t=0).** ***Initial values: Self-replacing without credit and debt positions and uniform rates of profits.***

	(1) Distri. $\mathbf{d}$	(2) Flow $\Delta\mathbf{CD}$	(3) Stock $\mathbf{F}^B$	(4) Prices $\mathbf{p}, w$	(5) Rates $\mathbf{r}, w$	(6) Barter Exp.	(7) Credit Exp.	(8) Barter Rev.	(9) Credit Rev.
Iron	0.18	0.00	0.00	11.52*	0.10	2.07	0.00	2.07	0.00
Coal	0.17	0.00	0.00	4.49*	0.10	2.02	0.00	2.02	0.00
Wheat	0.14	0.00	0.00	3.70*	0.10	1.77	0.00	1.77	0.00
Labour	0.51	0.00	0.00	0.51	0.51	0.51	0.00	0.51	0.00
Total	1.00	0.00	0.00	–	–	6.38	0.00	6.38	0.00

The triple $\mathbf{A}$, $\boldsymbol{\ell}$, $\mathbf{b}$ used for the computations was given in 4.1 (or PCMC, p. 19).
Numéraire: surplus vector, $\mathbf{s} = [0 \text{ t.iron}, 165 \text{ t.coal}, 70 \text{ qr.wheat}]^T$.
Columns: (1) Distribution of the surplus **s** to industries and workers; (2) Variations in credit and debt positions for the industries and for the workers (flows); (3) Credit and debt positions for the industries and for the workers at the end of the period (stocks); (4) Prices for the commodities produced by the individual industries and wage rate of the workers; (5) Profit rates and wage rates ; (6) Expenditures taking place as if they were barter exchanges (physical goods with physical goods, see section 3.3); (7) Expenditures taking place as if they were credit exchanges (physical goods exchanged now against deferred payments, IOUs, see above section 3.3); (8) Revenues taking place as if they were barter exchanges; (9) Revenues taking place as if they were credit exchanges
(*) Multiplied by $\times 10^{-3}$

means of payments becomes necessary for the system to be self-replacing. The associated prices and values of the new credit and debt contracts are also reported in table 4.2. We can see that both iron and coal industry are in debt, while the wheat industry and workers are in credit (see columns 2 and 3 and columns 7 and 9 of table 4.2, where the values are not any longer zeros as they were in table 4.1).

Once these debts are issued, we can analyse the means and duration of their repayment. There are several ways in which the debt extinction can be possible (yet keeping the system in a self-replacing state), for example, (a) through an abstention from consumption of the surplus in the subsequent period; (b) through a change in prices with constant consumption; (c) a combination of the two. For demonstration purposes, we consider case (a), while noting that the computation of other two cases are analogous, straightforward extensions. Similarly, the duration of repayment can vary with interest rate for servicing this debt.

Debt extinction through abstention from consumption

We explore the possibility of debt repayment by altering the quantity of consumption out of the surplus (i.e., the distribution) by different actors in the periods subsequent to the emergence of credit and debt. In this example, we keep the self-replacing prices and the wage/profit rates of period 1 (i.e., when credit and debt emerged) unaltered for the subsequent periods.[24] That is, the prices, wage rate and profit rates remain the same as in column (4-5) of 4.2 as the debt is repaid in the periods to follow. Because the prices do not change, the only possibility is to repay the debt is to abstain from consumption (a change in distribution **d** in column (1)) compared to previous levels. We assume that

[24] This is to keep the demonstration relatively straightforward and tractable. It is possible to experiment with alternative distribution structures, where in the actors choose their consumption levels in the different periods to follow. The associated new self-replacing prices, if they exist, will be different. This, in turn, has implications for the duration of repayment.

the interest rates are zero and we relax this assumption to explore the impact of changing interest rates on repayment times in the following subsection.

In our example, the industries which are in debt abstain from consumption out of the surplus, while the share of surplus going to those in credit increases. The evolution of these different variables in subsequent periods ($t = 2 \dots 7$) is reported in table 4.3, until the whole debt is paid back. The case examined here has no consumption out of the surplus for owners of iron and coal industries in period 2 and until they pay back their entire debt. This can be seen as the lower bound on possible abstention, which in turn facilitates faster repayment. It is possible to explore other preferable paths of consumption as well. We can see that the iron producers are the last to repay their debt (which lasts until period 7) while the coal producers repay the debt after period 3 and can begin to consume a positive share of the surplus from period 4 onward. With the emergence of new prices, note that the rates of profits are not uniform anymore.

Time to debt extinction and interest rate

In the previous section and in tables 4.1, 4.2 and 4.3, we examined the debt repayment path and the associated sequence of adjustments, all the while assuming that the financial interest rate is zero. The time required for complete repayment of debts however will increase as the exogenous financial interest rate increases. It can increase to the point, for sufficiently high financial interest rates, where debt extinction may even become impossible.

We examine how the time to debt extinction varies with financial interest rate i^F. Figure 4.5 shows the relation between (positive) financial interest rates and time to debt extinction. In our example, the time to debt extinction increases more than proportionally, becoming asymptotic around 20%. Above this interest rate, it would become impossible to repay the debt in finite time, given the existing production structure. Alternatively, we can also compute the range of interest rates that are compatible with repayment paths if there is a fixed, finite time period within which the entire debt needs to be paid back. This is also quite straightforward within the framework developed in this paper.

5 Conclusion

In this paper, we extended the traditional Sraffian schemes by introducing the possibility of having deferred means of payments. This act of transferring purchasing power between actors, with a commitment to pay back at a future point in time, has an impact on possibilities of evolution for the system. More specifically, this enlarges the domain of prices that allow the system to replicate the existing production structure. It also highlights the role that money plays in creating flexibility and structural viability in the system. We provide an explicit characterisation of the prices and wage rates for self-replacement, for a given distribution. Our framework is more general and accommodates differential or non-uniform rate of profits across industries.

In Zambelli (2018b), the uniform rate of profits assumption is removed from the original structure outlined in PCMC. This assumption is shown to be a special case of differential rates of profits. We further extended this framework here by allowing for credit and debt (IOUs) and studied the conditions for the economic system to be in a self-replacing state. Throughout this paper, as in PCMC, there has been no change in the surplus produced, means of production and the methods of production. The self-replacement requirement has been maintained throughout. We have also shown that the substance of Sraffa's critique concerning indeterminacy remains relevant even when there is credit,

Table 4.2: *Period 1 (t=1). Self-replacing with changed prices with respect to Period 0. Same distribution of the surplus as in period 0 and consequent emergence of credit and debt positions.*

	(1) Distri. $\mathbf{d}$	(2) Flow $\Delta\mathbf{CD}$	(3) Stock $\mathbf{F}^B$	(4) Prices $\mathbf{p}, w$	(5) Rates $\mathbf{r}, w$	(6) Barter Exp.	(7) Credit Exp.	(8) Barter Rev.	(9) Credit Rev.
Iron	0.18	-0.15	-0.15	10.21*	0.02	1.34	0.65	1.34	0.50
Coal	0.17	-0.10	-0.10	4.32*	0.04	1.51	0.53	1.51	0.43
Wheat	0.14	0.10	0.10	4.11*	0.17	1.56	0.31	1.56	0.41
Labour	0.51	0.15	0.15	0.66	0.66	0.51	0.00	0.51	0.15
Total	1.00	0.00	0.00	—	—	4.92	1.49	4.92	1.49

The triple $\mathbf{A}$, $\boldsymbol{\ell}$, $\mathbf{b}$ used for the computations was given in 4.1 (or PCMC, p. 19).
Numéraire: surplus vector, $\mathbf{s} = [0$ t.iron, 165 t.coal, 70 qr.wheat$]^T$.
Columns: (1) Distribution of the surplus $\mathbf{s}$ to industries and workers; (2) Variations in credit and debt positions for the industries and for the workers (flows); (3) Credit and debt positions for the industries and for the workers at the end of the period (stocks); (4) Prices for the commodities produced by the individual industries and wage rate of the workers; (5) Profit rates and wage rates ; (6) Expenditures taking place as if they were barter exchanges (physical goods with physical goods, see section 3.3); (7) Expenditures taking place as if they were credit exchanges (physical goods exchanged now against deferred payments, IOUs, see above section 3.3); (8) Revenues taking place as if they were barter exchanges; (9) Revenues taking place as if they were credit exchanges
(*) Multiplied by $\times 10^{-3}$

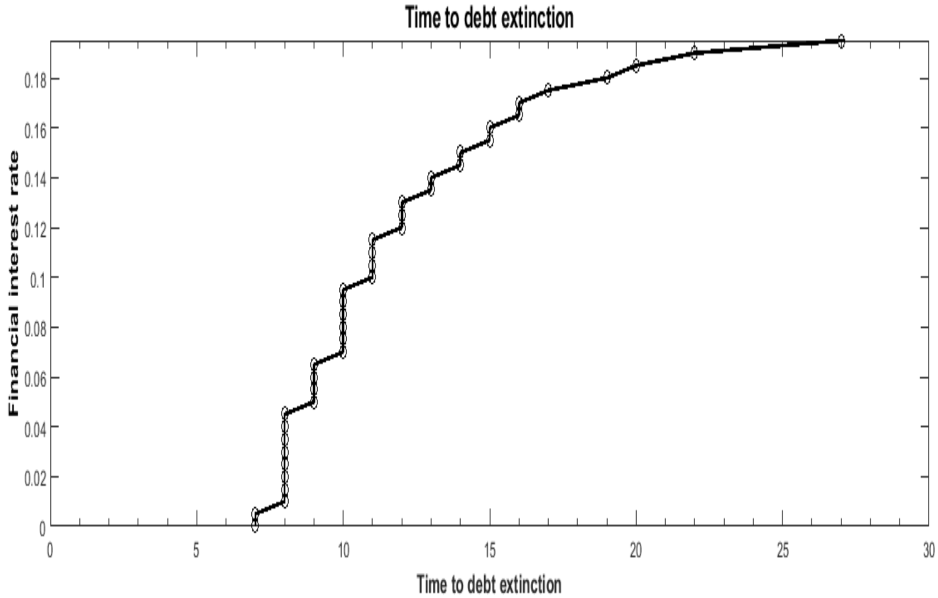

Figure 4.5: ***Time to debt extinction and the financial interest rate.*** *The time to debt extinction is different when the financial interest rate is different, given the debt and credit positions and the associated new financial balances described above when the prices change from those of table 4.1 to those of table 4.2.*

Table 4.3: *Period $t = 2 \ldots 7$. Self-replacing in period t, with same prices as in Periods $1 \ldots t-1$ and partial repayment of the debt. Changed distribution of the surplus.*

t		Distri. $\mathbf{d}$	Flow $\Delta\mathbf{CD}$	Stock $\mathbf{F}^B$	Prices $\mathbf{p}, w$	Rates $\mathbf{r}, w$	Barter Exp.	Credit Exp.	Barter Rev.	Credit Rev.
2	Iron	0.00	0.03	-0.12	10.21*	0.02	1.81	0.00	1.81	0.03
	Coal	0.00	0.07	-0.03	4.32*	0.04	1.85	0.03	1.85	0.10
	Wheat	0.26	-0.02	0.08	4.11*	0.17	1.89	0.11	1.89	0.09
	Labour	0.74	-0.07	0.08	0.66	0.66	0.64	0.10	0.64	0.02
	Total	1.00	0.00	-0.00	—	—	6.18	0.23	6.18	0.23
3	Iron	0.00	0.03	-0.09	10.21*	0.02	1.81	0.00	1.81	0.03
	Coal	0.01	0.06	0.03	4.32*	0.04	1.87	0.02	1.87	0.08
	Wheat	0.28	-0.05	0.03	4.11*	0.17	1.89	0.13	1.89	0.08
	Labour	0.71	-0.05	0.03	0.66	0.66	0.64	0.07	0.64	0.02
	Total	1.00	0.00	-0.00	—	—	6.21	0.21	6.21	0.21
4	Iron	0.00	0.03	-0.06	10.21*	0.02	1.81	0.00	1.81	0.03
	Coal	0.08	-0.01	0.02	4.32*	0.04	1.89	0.06	1.89	0.05
	Wheat	0.25	-0.01	0.02	4.11*	0.17	1.91	0.07	1.91	0.06
	Labour	0.67	-0.01	0.02	0.66	0.66	0.65	0.03	0.65	0.02
	Total	1.00	0.00	-0.00	—	—	6.26	0.16	6.26	0.16
5	Iron	0.00	0.03	-0.03	10.21*	0.02	1.81	0.00	1.81	0.03
	Coal	0.08	-0.01	0.01	4.32*	0.04	1.89	0.06	1.89	0.05
	Wheat	0.25	-0.01	0.01	4.11*	0.17	1.91	0.07	1.91	0.06
	Labour	0.67	-0.01	0.01	0.66	0.66	0.65	0.03	0.65	0.02
	Total	1.00	0.00	0.00	—	—	6.26	0.16	6.26	0.16
6	Iron	0.00	0.03	0.00	10.21*	0.02	1.81	0.00	1.81	0.03
	Coal	0.08	-0.01	0.00	4.32*	0.04	1.89	0.06	1.89	0.05
	Wheat	0.25	-0.01	0.00	4.11*	0.17	1.91	0.07	1.91	0.06
	Labour	0.67	-0.01	0.00	0.66	0.66	0.65	0.03	0.65	0.02
	Total	1.00	0.00	0.00	—	—	6.26	0.16	6.26	0.16
7	Iron	0.03	0.00	0.00	10.21*	0.02	1.84	0.00	1.84	0.00
	Coal	0.07	0.00	0.00	4.32*	0.04	1.94	0.00	1.94	0.00
	Wheat	0.24	0.00	0.00	4.11*	0.17	1.97	0.00	1.97	0.00
	Labour	0.66	0.00	0.00	0.66	0.66	0.66	0.00	0.66	0.00
	Total	1.00	0.00	0.00	—	—	6.42	0.00	6.42	0.00

The triple $\mathbf{A}$, $\boldsymbol{\ell}$, $\mathbf{b}$ used for the computations was given in 4.1 (or PCMC, p. 19).
Numéraire: surplus vector, $\mathbf{s} = [0 \text{ t.iron}, 165 \text{ t.coal}, 70 \text{ qr.wheat}]^T$.
Columns: (1) Distribution of the surplus $\mathbf{s}$ to industries and workers; (2) Variations in credit and debt positions for the industries and for the workers (flows); (3) Credit and debt positions for the industries and for the workers at the end of the period (stocks); (4) Prices for the commodities produced by the individual industries and wage rate of the workers; (5) Profit rates and wage rates ; (6) Expenditures taking place as if they were barter exchanges (physical goods with physical goods, see section 3.3); (7) Expenditures taking place as if they were credit exchanges (physical goods exchanged now against deferred payments, IOUs, see above section 3.3); (8) Revenues taking place as if they were barter exchanges; (9) Revenues taking place as if they were credit exchanges
(*) Multiplied by $\times 10^{-3}$

debt and non-uniform rates of profit. We believe that these extensions help provide the necessary ingredients towards developing a more general Sraffian monetary theory of production and distribution.

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References

Ciccarone, G. (1998). Prices and distribution in a Sraffian credit economy. *Review of Political Economy 10*(4), 399–413.

Fisher, I. (1911). *The Purchasing Power of Money. Its Determination and Relation to Credit Interest and Crises* (Second Revised Edition, 1920 ed.). London: The Macmillan Company.

Gilibert, G. (2003). The equations unveiled: Sraffa's price equations in the making. *Contributions to Political Economy 22*(1), 27–50.

Gilibert, G. (2006). The man from the moon: Sraffa's upside-down approach to the theory of value. *Contributions to Political Economy 25*(1), 35–48.

Panico, C. (1988). Sraffa on money and banking. *Cambridge Journal of Economics 12*(1), 7–28.

Pivetti, M. (1991). *An Essay on Money and Distribution.* London: Palgrave MacMillan.

Sraffa, P. *L'inflazione monetaria in Italia prima e dopo la guerra.* Premiata Scuola Tipografica Salesiana.

Sraffa, P. (1922a). The bank crisis in Italy. *The Economic Journal 32*(126), 178–197.

Sraffa, P. (1922b). Italian banking today. *The Manchester Guardian Supplement, Reconstruction in Europe, , Supplement, 7 December.*

Sraffa, P. (1932). Dr. Hayek on money and capital. *The Economic Journal 42*(165), 42–53.

Sraffa, P. (1960). *Production of Commodities by Means of Commodities.* Cambridge: Cambridge University Press.

Venkatachalam, R. and S. Zambelli (2021). Sraffa, money and distribution. In A. Sinha (Ed.), *A Reflection on Sraffa's Revolution in Economic Theory*, Chapter 15, pp. 455–490. London: Palgrave-Macmillan.

Zambelli, S. (2018a). The aggregate production function is NOT neoclassical. *Cambridge Journal of Economics 42*(2), 383–426.

Zambelli, S. (2018b). Production of Commodities by Means of Commodities and non-uniform rates of profits. *Metroeconomica 69*(4), 791–818.