ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/rquf20

Heterogeneity and clustering of defaults

A. K. Karlis, G. Galanis, S. Terovitis & M. S. Turner

To cite this article: A. K. Karlis, G. Galanis, S. Terovitis & M. S. Turner (2021): Heterogeneity and clustering of defaults, Quantitative Finance, DOI: 10.1080/14697688.2021.1882686

To link to this article: https://doi.org/10.1080/14697688.2021.1882686

Published online: 30 Mar 2021.

Quantitative Finance, 2021
https://doi.org/10.1080/14697688.2021.1882686

Heterogeneity and clustering of defaults

A. K. KARLIS†, G. GALANIS‡, S. TEROVITIS *§ and M. S. TURNER¶

†Department of Accounting, Economics and Finance, University of the West of England, Bristol, UK
‡Institute of Management Studies, Goldsmiths, University of London, London, UK
§Amsterdam Business School, University of Amsterdam, Amsterdam, The Netherlands
¶Department of Physics and Centre for Complexity Science, University of Warwick, Coventry, UK

(*Received 10 April 2019; accepted 8 January 2021; published online 30 March 2021*)

This paper studies how the degree of heterogeneity among hedge funds' demand orders for a risky asset affects the possibility of their defaults being clustered. We find that fire-sales caused by margin calls is a necessary, yet not a sufficient condition for defaults to be clustered. We show that when the degree of heterogeneity is sufficiently high, poorly performing HFs are able to obtain a higher than usual market share, which leads to an improvement of their performance. Consequently, their survival time is prolonged, increasing the probability of them remaining in operation until the downturn of the next leverage cycle. This leads to an increase in the probability of poorly and high-performing hedge funds to default in sync at a later time, and thus also in the probability of collective defaults. Our analytical results establish a connection between the nontrivial aggregate statistics and the presence of infinite memory in the process governing the hedge funds' defaults.

Keywords: Long memory; Financial crises; Hedge funds; Survival statistics; Bankruptcy

JEL Classification: C22, G01, G23, G32, G33

1. Introduction

The hedge fund (HF) industry has experienced an explosive growth in recent years. The total size of the assets managed by HFs in 2018 was estimated at US$2.9 trillion (Statista 2019). Due to the increasing weight of HFs in the financial market, failures of HFs can pose a major threat to the stability of the global financial system. The default of a number of high profile HFs, such as LTCM and HFs owned by Bear Stearns (Haghani 2014), testifies to this.

At the same time, poor performance of HFs—the prelude to the failure of a HF—is empirically found to be strongly correlated across HFs (Boyson *et al.* 2010), a phenomenon known as 'contagion'. Moreover, Boyson *et al.* (2010) point out that the correlation between HFs' worst returns—falling in the bottom 10% of a HF style's monthly returns—remains high, even after taking into account that HF returns are autocorrelated, and the effect of the exposure of HFs to commonly known risk factors. The findings of Boyson *et al.* (2010) support the theoretical predictions of Brunnermeier and Pedersen (2009), who provide a mechanism revealing how liquidity shocks can lead to downward liquidity spirals and thus to contagion.† The mechanism that leads to contagion is closely related to the theory of the 'leverage cycle', i.e. the pro-cyclical increase and decrease of leverage, due to the interplay between equity volatility and leverage, put forward by Geanakoplos (1997).‡

The combination of the dominant role of HFs in the financial system with the possibility of transmission of the risk, not only to other financial organisations but also to the real economy, has placed the operation of HFs under close scrutiny and has highlighted the significance of regulation of the industry. Regulating the HF industry is a challenging task; designing the appropriate regulation requires a good understanding of many aspects such as the mechanism which generates defaults at the individual level, the mechanism behind contagion, and finally the parameters which determine the persistency of the effect of a default of an individual HF on the industry. Although Brunnermeier and Pedersen (2009) provide the mechanism behind contagion, they overlook the persistency

*Corresponding author. Email: S.Terovitis@uva.nl

† Other works which study the causes of contagion in financial markets include Kyle and Xiong (2001) and Kodres and Pritsker (2002).
‡ In fact the theory of leverage cycle, in contrast to other models that endogenise leverage Brunnermeier and Pedersen 2009, Brunnermeier and Sannikov 2014, Vayanos and Wang 2012 has the additional merit of making the endogenous determination of collateral possible.

of the impact of a default of an individual HF. Our paper aims to fill this gap. In particular, we characterise the conditions under which the correlation between HF's defaults is persistent, i.e. defaults are clustered.

We study an economy with heterogeneous interacting agents (HIA)—HFs in our case—in the tradition of Day and Huang (1990), Brock and LeBaron (1996), Brock and Hommes (1997, 1998), Chiarella and He (2002), Thurner *et al.* (2012) and Poledna *et al.* (2014) among others.† We find that the feedback between market volatility and margin requirements (downward liquidity spiral) is a necessary yet not sufficient condition for clustering of defaults to occur, as has been suggested by Boyson *et al.* (2010). In this work, we show that heterogeneity plays a pivotal role in the emergence of clustered defaults: defaults are clustered only if the degree of heterogeneity is sufficiently high.

We develop a simple dynamic model with a finite number of HF managers and a representative mean-reverting noise trader trading a risky asset. We allow for a setup where heterogeneity regarding the demand of the risky asset may be due to different preferences towards risk, disagreement on the expected price of the asset, or disagreement on the volatility of the market. Also, we allow for the HFs to have access to credit, by using their existing assets as collateral. Finally, we endogenise the probability of default by assuming that an HF is forced to default when its portfolio value falls below a threshold.

In this environment, we show that when the degree of heterogeneity is sufficiently high, poorly performing HFs are able to absorb shocks caused by fire sales. As a result, they obtain a larger than usual market share and improve their performance. In this fashion, a default due to exactly their poor performance is delayed, allowing them to remain in operation until the downturn of the next leverage cycle. This leads to an increase in the probability of poorly and high-performing hedge funds to default in sync at a later time, and thus the probability of collective defaults. Formally, we show that for high degree of heterogeneity the default time-sequence shows infinite memory.

Using the definition of Andersen and Bollerslev (1997), clustering is determined by the divergence of the sum (or integral in continuous time) of the autocorrelation function (ACF) of the default time sequence. Therefore, the underlying stochastic process describing the occurrence of defaults exhibits long memory. We establish a quantitative connection between non-trivial aggregate statistics and the presence of infinite memory in the underlying stochastic process governing defaults of the HFs. The comparison between the theoretical prediction of the asymptotic behaviour of the autocorrelation function (ACF) of defaults and the numerical findings, reveals that our theoretical predictions are valid even in a market with a finite number of HFs and the clustering of defaults is confirmed. In this way, our model provides a novel insight into the empirical findings of Boyson *et al.* (2010), which highlight the role of heterogeneity.

Our paper makes also a methodological contribution to the related literature as our analysis combines analytic and numerical methods in a novel way.

† For a detailed relevant literature review, see Hommes 2006, LeBaron 2006 and Chiarella *et al.* 2009.

The structure of the rest of the paper is as follows. Section 2 discusses the relevant literature. Section 3 presents the economic framework that we use. Section 4 discusses the numerical findings and also provides analytical results linking the heavy-tailed aggregate density to the observed statistical character of defaults at a microscopic level, and the power-law decay of the ACF of the default time-series. Finally, section 5 provides a short summary with concluding remarks.

2. Relevant literature

Our paper is related methodologically to the HIA literature; and in terms of content, to the literature which studies the effects of leverage on financial stability.‡ While the vast majority of the papers in the HIA literature rely mainly in computational methods, the present paper combines computational methods with analytical results in a novel way which allows for better economic insights. Even though our analysis is methodologically original, the recent papers of Chiarella and Di Guilmi (2011), Di Guilmi and Carvalho (2017) and He *et al.* (2017) are also good examples where analytical tools are used in combination to computational methods.

Models with HIA can give rise to emergent properties of systems that are able to replicate the empirical trends seen in asset prices, asset returns and their distributions (Lux 1995, 1998, Lux and Marchesi 1999, Iori 2002, He and Li 2007, Chiarella *et al.* 2014). In Levy (2008), spontaneous crashes are a natural property of a market with heterogeneous investors who are inclined to conform to their peers, under the condition that the strength of the conformity effects is large compared to the degree of heterogeneity of the investors. In other papers, such as Chiarella (1992), Lux (1995) and Di Guilmi *et al.* (2014) heterogeneity has to do with the different beliefs and trading rules of the agents (fundamentalists and chartists) which can result to asset price fluctuations and market instability.

The set up of our model is similar to Thurner *et al.* (2012) and Poledna *et al.* (2014) which study the effects of leverage in an economy with heterogeneous HFs. Thurner *et al.* (2012) show that leverage causes fat tails and clustered volatility. Under benign market conditions, HFs become more leveraged as this is then more profitable. High levels of leverage are correlated with increased asset price fluctuations that become heavy-tailed. The heavy tails are caused by the fact that when an HF reaches its maximum leverage limit then it has to repay part of its loan by selling some of its assets. Poledna *et al.* (2014) use a very similar framework to test three regulatory policies: (i) imposing limits on the maximum leverage, (ii) similar to the Basle II regulations, and (iii) a hypothetical perfect hedging scheme, in which the banks hedge against the leverage-induced risk using options. They find that the effectiveness of the policies depends on the levels of leverage, and that even though the perfect hedging scheme reduces volatility in comparison to the Basle II scheme, none of these are able to make the system considerably safer on a systemic level.

‡ The present paper focuses on the role of leverage on a microeconomic level and does not discuss the feedback effects with the Macroeconomy. For the latter, see Chiarella and Di Guilmi 2011, Ryoo 2010 and references therein.

Our model extends this framework in two directions. First, in our model the behaviour of HFs is not given by heuristics but it is derived from first principles. In both Thurner *et al.* (2012) and Poledna *et al.* (2014), HFs are risk neutral and have different demand of the asset given the same information and the same wealth. The characteristic which makes them heterogeneous is called ‘aggression’ and aims to capture different responses of the agents to a mispricing signal. Given the risk neutrality assumption, it is impossible to provide a rigorous explanation for the difference in aggression. Furthermore, deriving the HFs demand functions from first principles: (i) we bridge the gap between Thurner *et al.* (2012) and Poledna *et al.* (2014); and the rest of the leverage cycle literature discussed below and (ii) we provide a framework which allows the study of different types of heterogeneity.

The leverage cycle models start with the collateral equilibrium models of Geanakoplos (1997) and Geanakoplos and Zame (1997), who provide a general equilibrium model of collateral. The key idea behind these models is that lenders require a collateral from the borrowers in order to lend them funds. This borrowing and lending is agreed through a contract of a promise of paying back the loan in future states, where the investor who sells the contract is borrowing money—using a collateral to back the promise—from the agent who buys the contract. Each contract is chosen from a menu of contracts with different loan to value (LTV) ratio. In Geanakoplos (1997), scarcity of collateral leads to only a few contracts being traded, which makes leverage (LTV) endogenous. Finally, the investors default when the value of the collateral is less than the value of the contract that borrowers and lenders have agreed. Geanakoplos (2003) considers a continuum of risk neutral agents with different priors in a binomial economy with two or three states of the world. He shows how changes in volatility lead to changes in equilibrium leverage which in turn have a bigger effect in asset prices than what agents believe to be the effect of news. Geanakoplos (2003, 1997) show that in some cases all agents will choose the same contract from the contract menu. This result has been recently extended by Fostel and Geanakoplos (2015) who study in more detail the relationship between leverage and default and prove that in all binomial economies with financial assets, exactly one contract is chosen.

Fostel and Geanakoplos (2008) extend the economy of Geanakoplos (2003) to an economy with multiple assets and two risk averse agents instead of a continuum of risk neutral ones and develop an asset pricing theory which links collateral and liquidity to asset prices. Geanakoplos (2010) combines the insights from Geanakoplos (1997) where the collateral is based on non financial assets and Geanakoplos (2003) where the collateral is based on financial assets; and shows that the introduction of CDS contracts reduces the asset prices. By doing this he puts forward a model of a *double leverage cycle*, in housing and securities, which contributes in the explanation of the 2007–08 crisis. Fostel and Geanakoplos (2012) provide a further analysis of CDS contracts and show: (i) why tranching and leverage initially raised asset prices and (ii) why CDSs lowered them later. Simsek (2013a) considers a continuum of states and two types of agents beliefs, namely optimist and pessimist. He shows that the type of disagreement between agents has more important effects on asset prices than the degree of disagreement between optimists and pessimists.† To our knowledge, this is the only paper in this literature which considers the effect of different degrees of heterogeneity.‡

Along similar lines the effects of leverage have been studied by Gromb and Vayanos (2002), Acharya and Viswanathan (2011), Brunnermeier and Pedersen (2009), Brunnermeier and Sannikov (2014) and Adrian and Shin (2010), among others. These approaches differ from the models mentioned in the previous paragraphs in two key aspects. The models of Acharya and Viswanathan (2011), Adrian and Shin (2010), Brunnermeier and Sannikov (2014) and Gromb and Vayanos (2002) focus on the ratio of an agent’s total asset value to his total wealth (investor based leverage) while the leverage cycle models of Geanakoplos and coauthors§ focus on LTV. The second aspect has to do with the fact that in the models of Brunnermeier and Pedersen (2009) and Gromb and Vayanos (2002) the leverage ratio is exogenously given, where in the former is given by a *VaR* rule, whereas in the latter it is given by a maximin rule used to prevent defaults. In the cases of Brunnermeier and Sannikov (2014), Acharya and Viswanathan (2011) and Adrian and Shin (2010) leverage is endogenous but is not determined by collateral capacities. In Acharya and Viswanathan (2011) and Adrian and Shin (2010) leverage is determined by asymmetric information between borrowers and lenders, while in Brunnermeier and Sannikov (2014) it is determined by agents’ risk aversion.

3. Model

3.1. Environment

We study an economy with two types of assets: a riskless asset, which can be interpreted as cash C, and a risky asset. As opposed to the riskless asset, which is in infinite supply, the supply of the risky asset is fixed and equal to N. The price of the riskless asset is normalised to 1, whereas the price of the risky asset at time t, p_t, is determined endogenously. Both types of assets are traded by the portfolio managers of each of the K hedge funds (HFs), and by a representative, mean-reverting noise trader. Finally, the model consists of an infinitely liquid bank, whose role is to provide credit to HFs, by using their assets as collateral.

Hedge Fund Managers. Each HF $j \in \{1, \ldots, K\}$ is run by a portfolio manager whose task is to submit the trading orders for the risky and the riskless asset, where D_t^j (C_t^j) denotes the units of the risky (riskless) asset the manager of HF j is willing to trade at time t. We assume that the compensation of the manager in period t is a fixed fraction γ of the HF’s net

† Other works in the leverage cycle literature include Geanakoplos and Zame (2014), Geanakoplos (2014) and Fostel and Geanakoplos (2016). For a recent review of this literature, see Fostel and Geanakoplos (2014).

‡ In a different context, Simsek (2013b) shows that the level of belief disagreement affects the average consumption risks of individuals in a model which studies the effect of financial innovation on portfolio risks.

§ Also the models of Brunnermeier and Pedersen (2009) and Simsek (2013a) use the same ratio.

asset value (NAV) at t, W_t, regardless of a HF's performance.† Moreover, we assume that the managers are myopic, thus, the objective of the manager of HF j is to maximise the next period's expected utility,

$$\mathbb{E}\left[U(\gamma W_{t+1}^j)|I^j\right],$$

where I^j denotes the information set of the manager of HF j. We focus on the case where managers have Constant Relative Risk Aversion (CRRA) preferences, i.e.

$$U\left(\gamma W_{t+1}^j\right) = \gamma {W_{t+1}^j}^{1-a} / (1-a), \tag{1}$$

where $a > 0$ is the measure of relative risk aversion. Given the manager's compensation method, only part $(1-\gamma)$ of the wealth of the HF is available for re-investment in the next period. This feature also excludes unrealistic cases where the wealth of HFs explodes and default never occurs.‡ Taking this into account, the wealth of an HF evolves according to:

$$W_{t+1}^j = (1-\gamma)\left[W_t^j + (p_{t+1} - p_t)D_t^j\right], \tag{2}$$

where the first term of the RHS captures the value of the portfolio held in the previous period and the second term captures the change in the value of the risky assets.

Hedge Funds' access to credit. A fundamental component of the Hedge Fund industry is the use of leverage. To capture this aspect, we allow HF to have access to credit. As a result, the amount of cash required to complement the trading order for a risky asset, i.e.$D_t^j p_t$, may exceed the cash which is available at the beginning of each trading period. However, the access to credit is not unbounded: the HF cannot become more leveraged than $\lambda_{\max}$. Thus $\lambda_{\max}$, which can be subject to regulation, is a highest feasible ratio of the market value of the risky asset held as collateral by the bank to the net wealth of the risky asset, i.e.

$$D_t^j p_t / W_t^j \le \lambda_{\max}.$$

Consequently, the maximum demand for the risky asset is given by

$$D_{t,\max} = \lambda_{\max} W_t^j / p_t, \ \forall j \in \{1, \ldots, K\}. \tag{3}$$

Furthermore, we allow the HFs to take only long positions, i.e. to be active only when the asset is underpriced.§

Hedge Funds' default. The main objective of this paper is to study both the individual (HF) and collective (systemic) default probabilities over time. To this end, we endogenise the probability of default of each HF, by requiring that an HF is forced to default when its wealth falls below $W_{\min} \ll W_0$, where W_0 denotes the initial endowment of each HF upon entrance in the market. In order to maintain the character of the market (at a statistical level), $T_r \sim \mathrm{U}[b, c]$ time-steps after the default, the bankrupt HF is replaced by an HF with identical characteristics.

Noise traders. The second type of traders is noise-traders, who are supposed to trade for liquidity reasons. In order to be consistent with related literature, we assume that the demand d^{nt} of the representative noise-trader for the risky asset, in terms of cash value, is assumed to follow a first-order autoregressive [AR(1)] process (Xiong 2001, Thurner *et al.* 2012, Poledna *et al.* 2014).

$$\log d_t^{\mathrm{nt}} = \rho \log d_{t-1}^{nt} + (1-\rho)\log(VN) + \chi_t, \tag{4}$$

where $\rho \in (0, 1)$ is a parameter controlling the rate of reverting to the mean. Given that the expected value of χ_t and the auto-covariance function are time independent, the stochastic process is wide-sense stationary, $\chi_t \sim \mathcal{N}(0, \sigma_{\mathrm{nt}}^2)$, and V is the fundamental value of the risky asset.¶

Equilibrium Condition. Finally, the price of the risky asset is determined endogenously by the market clearing condition [together with equations (2), (4), and (7)].‖

$$D_{t+1}^{\mathrm{nt}}(p_{t+1}) + \sum_{j=1}^{K} D_{t+1}^j(p_{t+1}) = N, \tag{5}$$

where $D_{t+1}^{\mathrm{nt}}(p_{t+1}) = d_{t+1}^{\mathrm{nt}}/p_{t+1}$ stands for the demand of the noise traders whereas $D_{t+1}^j(p_{t+1})$ stands for the demand of the HF j. Both values are in number of shares.

Timing. Each period t consists of four sub-periods.

(i) The managers set their demand orders for the risky asset.
(ii) The price of the risky asset is determined, and the return of each portfolio is realised.
(iii) The managers receive their compensation.
(iv) The next-period's wealth is determined.

3.2. *Optimal demand*

The manager of the HF j maximises her expected utility, given her beliefs I^j about the asset's fundamental value and the volatility of the market, and subject to the constraint that the demand cannot exceed $D_{t,max}^j$.

$$D_t^j = \mathrm{argmax}_{D_t^j \in [0, D_{t,\max}]}\left\{\mathbb{E}\left[U(\gamma W_{t+1}^j)|I^j\right]\right\} \tag{6}$$

† The management fee is typically around 2% of a HF's net asset value Stowell (2010, p. 199).

‡ Considering only the management fee which is a time-independent fraction γ of an HF's NAV and ignoring the performance fee that has a more complicated structure allows us to develop a more tractable model. However, the critical component for our main findings presented below is the existence of a mechanism that prohibits the consistent (on average) flow of capital from the NTs to the HFs. The latter ensures that statistical averages in time are well-defined.

§ We do this in order to highlight that, even with the HFs taking only long positions, a strategy inherently less risky than short-selling, the clustering of defaults, and thus systemic risk, is still present if heterogeneity among the prior beliefs is sufficiently large.

¶ The demand of the noise traders in terms of the number of shares of the risky asset D^{nt} and the price of the risky asset p_t at period t is $d_t^{nt} = D_t^{nt} p_t$. Hence, in the absence of the HFs, from equation (4), and equation (5) we have $\mathbb{E}[\log p_{t+1}] = \log V$.

‖ This system of equations is highly non-linear, and thus, can only be solved numerically.

Solving the optimisation problem we obtain†

$$D_t^j = \min\left\{\frac{1}{a}\left(s_j \log\left(V/p_t\right) + \frac{1}{2}\right), \lambda_{\max}\right\}\frac{W_t^j}{p_t}, \qquad (7)$$

where $s_j = 1/\mathrm{Var}[\log p_{t+1}|I^j]$. Therefore, the demand of the HFs is proportional to the expected logarithmic return and their wealth, and inversely proportional to the conditional variance of the logarithm of the price, given their beliefs.

3.3. Source of heterogeneity

A critical component, which lies at the heart of our analysis, is heterogeneity across HFs. We allow for a setup where different HFs respond differently when facing the same price. In particular, we develop a model where, for a give price p_t, different HFs will find it optimal to post different demand orders for the risky asset, i.e. $D_t^i \neq D_t^j$ for $i \neq j$. One can think of many reasons which could justify heterogeneity across HFs. One explanation could be that HFs have different beliefs about the fundamental value V of the asset. Another reason which could justify this heterogeneity could be that HFs agree on the mean, but the disagree on the variance, i.e. $\mathrm{Var}[\log p_{t+1}|I^j]$. Finally, HFs' heterogeneity might be driven by different degrees of risk aversion, i.e. a. The main findings are qualitatively equivalent independently of which of the previous possible interpretations is implemented. Throughout the paper, we assume that HFs disagree on the market volatility.

The rationale behind the assumption that the managers agree on the fundamental value of the asset, but disagree on price volatility, relies on the fact that the fundamental value, as opposed to price volatility, is not affected by the behaviour of HFs. In other words, the fundamental value of the asset is exogenously determined, whereas the volatility of the market is endogenously determined, with its value depending on the HFs' trading strategy, which in turn, depends on their private information set. Hence, it is not feasible for the managers to reach an agreement on the market volatility, because they have access to different information sets, and the market volatility is affected by the information each manager has access to.

Our main goal is to study the relationship between the degree of heterogeneity κ and clustering of defaults, where κ is identified with the difference between the maximum and minimum values of s_j in the demand order (7). The question arises as to whether the leverage cycle‡ is a sufficient condition for the defaults to be clustered, or rather whether there exists a critical value for the degree of heterogeneity above which the mechanism of the leverage cycle leads to clustering of defaults.

3.4. Clustering of defaults

The clustering of HFs' defaults is determined by the decay rate of the default time-sequence autocorrelation function (ACF) $C(t')$, with t' being the time-lag variable. If defaults are clustered, then $C(t')$ decays in such a way that the sum of the ACF over the lag variable diverges (Baillie 1996, Samorodnitsky 2007).

Definition 1 *Let $C(t')$ denote the autocorrelation of the time series of defaults, with t' being the lag variable. Defaults are clustered if and only if*

$$\sum_{t'=0}^{\infty} C(t') \to \infty. \qquad (8)$$

Given that the ACF is bounded in $[-1, 1]$, it follows that the convergence of the infinite sum is in turn determined by the asymptotic behaviour $t' \gg 1$ of the ACF. In this limit, the sum can be approximated by an integral. In the following we assume that the ACF of the default time sequence can be approximated by a continuous function for $t' \gg 1$. Then it follows that,

Remark 1 Defaults are clustered if the ACF asymptotically approaches zero not faster than $C(t') \sim 1/t'$. In this case, defaults are interrelated (statistically dependent) for all times.

Remark 2 If the decay of the ACF is faster than algebraic, then defaults are not clustered. The effect of the shock caused by the default of a HF on the market is only transient, and the defaults are in the long-run statistically independent.

In the next section, we present the results of the model. The first subsection presents the numerical results obtained by iterating the model defined above. We present the ACFs for various values of κ and interpret these in light of Remarks 1 and 2. Section 4.2 provides an analytical insight into the numerical results.

3.5. Discussion of the main assumptions

Before we explore which assumptions are essential, it is important to present the underlying mechanism behind the main findings. As it becomes clear in the next section, our work shows that when the degree of heterogeneity is sufficiently high, poorly performing HFs are able to absorb shocks caused by fire sales. As a result, they obtain a larger than usual market share and improve their performance. In this fashion, a default due to exactly their poor performance is delayed, allowing them to remain in operation until the downturn of the next leverage cycle. This leads to an increase in the probability of poorly and high-performing hedge funds to default in sync at a later time, and thus the probability of collective defaults.

The mechanism described in the previous paragraph implies that the key components of our model are: (i) the leverage cycle; (ii) heterogeneity in the managers' action, and (iii) risk aversion. In fact, the risk aversion assumption is not essential to generate the main findings. All that matters is there is some heterogeneity in the way managers invest, i.e. given the price of the asset, different managers post different trading orders. However, allowing for risk aversion enables us to provide micro-foundations of the managers' heterogeneity; when the managers observe different private signals about the fundamental value of the asset, their optimal demand order will differ.

† For details, see Appendix 1.
‡ Recall that leverage cycle is defined as the pro-cyclical increase and decrease of leverage, due to the interplay between equity volatility and leverage.

Simulating a model which (i) allows the managers to optimally determine their demand orders and (ii) adopts preferences and economic parameters which find strong empirical support, enables us to shed light on why and when we would expect to observe clustering of defaults. Knowing why and when the defaults will be clustered has important implications for (i) the design of the optimal portfolio by investors and (ii) the design of policies by a regulator whose objective is to minimise the incidents of clustered defaults.

Both the analytical and the numerical part requires us to adopt a set of assumptions about the functional form of the managers' preferences. To this end, we assume CRRA preferences, which is a standard assumption in the portfolio allocation (see Danthine and Donaldson 2005 for a discussion of preferences), and it is also supported by the empirical evidence by Szpiro (1986) and Szpiro and Outreville (1988). Focusing on CRRA preferences (compared to, for instance, Constant Absolute Risk Aversion (CARA) preferences) has two desirable properties. First, CRRA preferences imply that the lower the HF's wealth, the higher the degree of risk aversion. This property is consistent with the idea that HF managers are relatively more cautious/conservative when the HF's wealth is low. The second useful property of CRRA is that, as opposed to CARA, the fraction of wealth invested in the risky asset under the optimal portfolio is independent of the level of initial wealth. Shutting down the wealth effects allows us to isolate the impact of HF managers' heterogeneity that goes above and beyond the income heterogeneity. Thus focusing on CRRA instead of CARA preferences allows us not only to develop a more realistic but also a more parsimonious model, that, in turn, enables us to isolate the impact of heterogeneity, which is the main scope of this paper.

4. Results

4.1. Numerical results

4.1.1. Choice of parameter values. In all simulations, we consider a market with $K = 10$ HFs. In the following, we assume homogeneous preferences towards risk across HFs, and set $a_j = 3.2\ \forall j \in \{1, \ldots, 10\}$, this being a typical value for HFs (Gregoriou *et al.* 2005, p. 417). From equation (4) we have $\tilde{\sigma}^2_{\rm nt} = \sigma^2_{\rm nt}/(1-\rho^2)$, where ρ is the mean reversion parameter. The inverse of the expected volatility given the HF's prior beliefs, i.e. $s_j = 1/{\rm Var}[\log p_{t+1}|I^j]$ determines the responsiveness of the HFs to the observed mispricing. In our numerical simulations s_j is sampled from a uniform distribution in $[1, \delta]$, and $\delta \in [1.2, 10]$. Moreover, the maximum allowed leverage $\lambda_{\max}$ is set to 5. This particular value is representative of the mean leverage across HFs employing different strategies (Ang *et al.* 2011). In order to be consistent with the findings of Poledna *et al.* (2014), we adopt the same parameters values, i.e. $\sigma^2_{\rm nt} = 0.035$, $V = 1$, $N = 10^9$, $W_0 = 2\times 10^6$, $W_{\min} = W_0/10$, $\rho = 0.99$ (Poledna *et al.* 2014), and $\gamma = 5\times 10^{-4}$.† Bankrupt HFs are reintroduced after T_r periods, randomly chosen according to a uniform distribution in $[10, 200]$.‡§ Furthermore, the HFs anticipate that their actions—buying when the risky asset is undervalued—will help moderate the fluctuations realised in the market. In other words, all HFs correctly believe that the volatility of the market will be reduced when they enter the market, in comparison to the volatility observed when only the noise traders are active. However, they are uncertain about their collective market power, and therefore the extent to which they will affect the realised volatility. Thus, all HFs believe that $\mathbb{E}[{\rm Var}(\log p_{t+1})|I^j] < \sigma^2_{\rm nt}/(1-\rho^2)$.¶

4.1.2. Simulations. As aforementioned, the leverage cycle consists of the interplay between the variability of prices of the assets put as collateral, and margin requirements. When prices are high, assets used as collateral are overpriced, and creditors are willing to lend. In the face of an abrupt fall of the market price of the assets used as collateral, creditors force the lenders to repay part of the loan, such that the margin requirements are met. Consequently, the lenders are forced to sell in a falling market, accelerating and reinforcing the fall of the price of the collateral, creating thus a vicious cycle. In our model, a fall in the price of the risky asset used as collateral is caused by a sudden drop of the demand of the noise traders d^{nt}_t. This results into a sudden increase of the leverage ratio of the jth HF, λ^j_t. In case λ^j_t exceeds the margin requirement $\lambda^j_t \leq \lambda^j_{\max}$ HFs are forced to sell, pushing the price even lower.

*Case with a low degree of heterogeneity.*We start by presenting the case with a low degree of heterogeneity ($\kappa = \delta - 1 = 0.5$), which is illustrated in figure 1. Figure 1 presents: (a) the wealth of three HFs (under, moderately, and highly responsive to mispricings, $j = 2, 6, 10$), (b) the corresponding leverage ratio, (c) the demand of the noise traders, and (d) the price of the risky asset at equilibrium as a function of time. At time $t = 738$ [marked by a blue triangle in panel (c)] a drop in the demand of the noise-traders causes an underpricing of the risky asset backing up the loans of HFs [panel (d)].

† The specific choice of γ maps 40 time-periods to a year considering that HF managers receive about 2% of the HF's net asset value under their management per annum. Given the set of values considered for all other parameters, this calibration of the model leads to an average rate of return of the HFs after the payment of the fees $\sim 6\%$ p.a. for a moderate degree of heterogeneity κ, which is a reasonable value.

‡ This corresponds to an average time of an HF manager being reintroduced into the market ~ 2.6 years after a default event given the calibration of the model, which seems a reasonable period of time. Furthermore, setting the minimum time b until an HF manager becomes active again to a positive value that maps to a quarter (in physical time) rules out the unrealistic possibility of a HF manager reappearing in the market immediately after the default of the HF under her management.

§ As this paragraph indicates, the set of parameter values are not chosen arbitrarily. All parameter values are chosen either according to estimations found in empirical studies Ait-Sahalia *et al.* 2004, Gregoriou *et al.* 2005, Stowell 2010, Ang *et al.* 2011 [for the parameters that can be estimated using empirical data] or in consonance with the relevant theoretical literature Poledna *et al.* 2014 [for those that cannot be directly estimated on data].

¶ A complete robustness analysis is not feasible because of (a) the large dimensionality of the parameter space and (b) most of the parameters are not mutually independent. For example, the parameters V, W_0, K, and N together determine the collective market power of the HFs compared to the NTs when they are introduced into the market. Therefore, none of these parameters can be set independently.

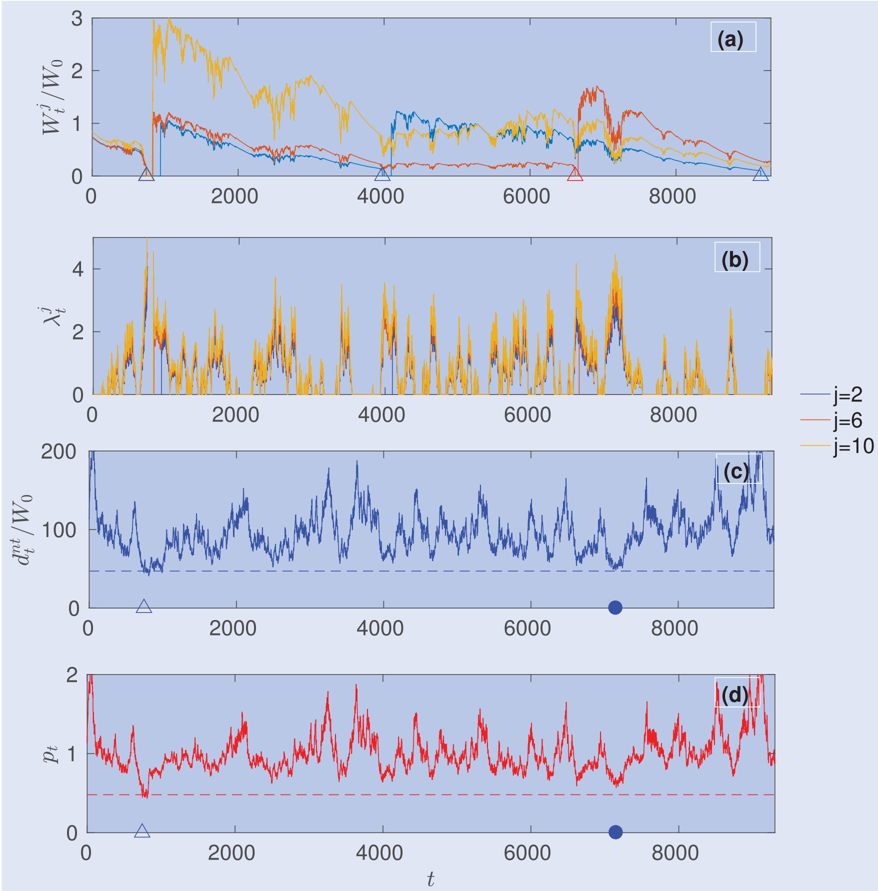

Figure 1. (a) The wealth normalised by the endowment W_0, (b) the leverage ratio λ_t^j, (c) the demand of the noise traders in terms of money-value, normalised also by W_0, and (d) the equilibrium price of the risky asset, as a function of time, in the case of $\kappa = 0.5$.

In turn, the leverage ratio of all the HFs depicted in figure 1(b) $\lambda_{t=738}$ increases abruptly [panel (b)], and the margin requirement $\lambda_{\max} = 5$ becomes binding for the most responsive of the HFs depicted ($j = 6, 10$). At this point, the HFs are forced to deleverage pushing the price of the collateral further down, leading all HFs depicted to default [panel (a)]. The pressure on the price of the risky asset due to the synchronous deleveraging of the highly responsive HFs can clearly be recognised if we compare the lowest price reached around the downturn of the leverage cycle at about $t = 738$ [marked by a the dashed red line in panel (d)], with the equilibrium price at $t = 7153$ [blue filled circle], where the demand of the noise trader becomes virtually the same to that at $t = 738$ [marked by a blue triangle], but the price remains at a considerably higher level. This is because the wealth of all HFs in this case, is such that the leverage ratio stays well below the maximum threshold [see panel (b)], and the leverage cycle mechanism remains inactive.

Another observation worth commenting on is the fact that after the HFs have been reintroduced in the market, we notice that the least responsive HF ($j = 2$), defaults another 2 times, by the end of the time-series depicted in figure 1, namely at $t = 3976$ and $t = 9161$ [also marked by blue triangles in panel (a)]. not because of the presence of a shock in the demand of the risky asset, but rather, due to its poor performance. This is because time is costly in our model (HFs pay managerial fees), and if the profitability of an HF is low, then it will inevitably be led to bankruptcy, even in the absence of a shock on the demand of the risky asset. These defaults happen at random times, i.e. when the observed mispricings happen to be small, or when the asset is overpriced, for a period of time, and the profits made are also small, or null, respectively. This also explains the second default of the sixth HF, at $t = 6618$ (red triangle in figure 1 a), when all the HFs are well below the maximum leverage constraint.

Case with a high degree of heterogeneity. We now explain the case with a higher degree of heterogeneity ($\kappa = 3$), which is illustrated in figure 2. Figure 2 presents the wealth W_t^j (panel a), the leverage ratio λ_t^j (panel b) of three representative HFs [$j = 2, 6, 10$], as well as the logarithmic returns (panel c) as a function of time. At $t = 493$ (marked by a red circle in panels a and c) the leverage cycle becomes active, causing an underpricing of the risky asset. However, the least responsive to mispricings hedge fund ($j = 2$) of the three depicted, manages to absorb the shock, as it stays below the maximum leverage $\lambda_{\max} = 5$ (see panel b, blue line), and never receives a margin call. However, the bankruptcy of the more responsive HFs, offers the HF that has survived the shock ($j = 2$), the opportunity to seize a larger market share and, as a result, to perform better in the short-run, restoring its wealth to a level similar to the one before the shock occurred. In this way, the most poorly performing HF is given the opportunity to continue operating until the next downturn of the leverage cycle, at which point it

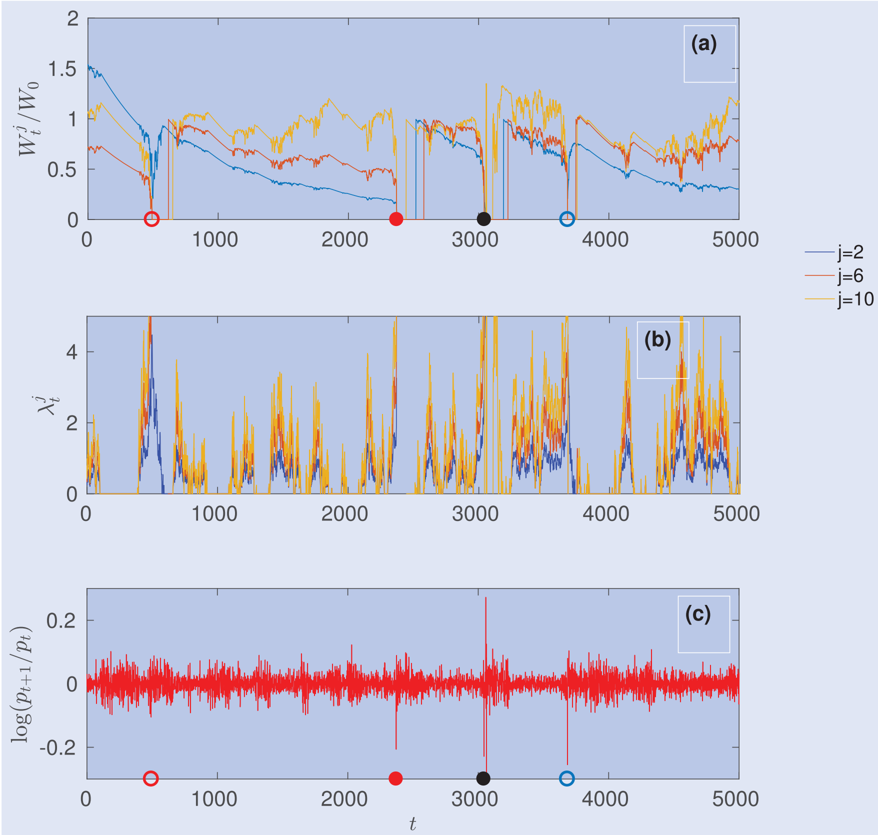

Figure 2. (a) The wealth normalised by the endowment W_0, (b) the leverage ratio λ_t^j, and (c) the logarithmic returns on the risky asset, as a function of time, with $\kappa = 3$.

defaults along with the rest of the HFs at $t = 2371$ (red disc). After the second crash of the market we observe the end of yet another leverage cycle, at which point all the depicted HFs default again in sync at $t = 3044$ (black disc). The narrative is repeated once more at $t = 3684$ (blue circle), when again the least responsive HF after absorbing the shock gets a larger market share, increasing shortly its profitability.

Low versus high degree of heterogeneity. In conclusion, the study of time-series in the case of low ($\kappa = 0.5$) and high heterogeneity ($\kappa = 3$) reveals that increased heterogeneity leads to the increase of collective defaults. Even more, the synchronous default of highly responsive HFs, gives the opportunity to the less responsive ones to increase their market share, and thus, their profitability, even for a short-period of time. Still, this increases the chance of the poor-performing HFs to survive until the next downturn of the leverage cycle, suppressing defaults occurring at random times due to their poor performance, and thus increasing even more the probability of synchronous defaults. Therefore, this analysis hints that the degree of heterogeneity is intimately connected to the level of systemic risk in the market.

To assess quantitatively the effect of the degree of heterogeneity, explained above, on the systemic risk, we study the persistence of the correlation between defaults (see Definition 1). In figure 3(a) we compare the numerically computed ACF of the default time-sequence† as observed on the aggregate level for 11 different degrees of heterogeneity κ, determined by the support of the distribution of s_j. The results were obtained by iterating the model described in section 3 for up to 3×10^8 periods, and averaging over 40 realisations of the responsiveness s_j; namely, $s_j \sim \mathrm{U}[1, \delta]$, with $\delta = \{1.2, 1.4, 1.7, 2, 3, 5, 6, \ldots, 10\}$. Clearly, when the degree of heterogeneity $\kappa \leq 1$, the ACF decays far more rapidly in comparison with larger values of heterogeneity. In fact, as it can be observed in the figure, the ACF for $\kappa \leq 1$ decays faster than a power-law with exponent equal to -1 (black dashed line), which is the largest exponent (in absolute terms) leading to a non-integrable ACF (see Remark 1). On the other hand, the converse is true for large degrees of heterogeneity ($\kappa > 2$), in which case the ACF decays asymptotically—$t' \gg 1$—as a power-law with exponent less than 1 in absolute value. Consequently,

RESULT 1 For $\kappa \leq 1$, the ACF decays faster than a power-law with exponent –1. Hence, the mechanism of the leverage cycle does not result into sufficiently high long-range correlations for defaults to be clustered.

Figure 3(a) also shows that for increasing heterogeneity the ACF converges to a limiting form as the heterogeneity is increased, which is reflected in the coalescence of the ACFs corresponding to $\kappa \geq 5$. The latter is more clearly demonstrated in figure 3(b), where a blow-up of the area within the rectangle shown in panel (a) is presented. Therefore,

RESULT 2 For sufficiently large values of the degree of heterogeneity κ, namely for $\kappa \geq 5$, the ACF converges to a

† The time-sequence considered is constructed by mapping defaults to 1s, irrespective of which HF defaulted, and to 0 otherwise.

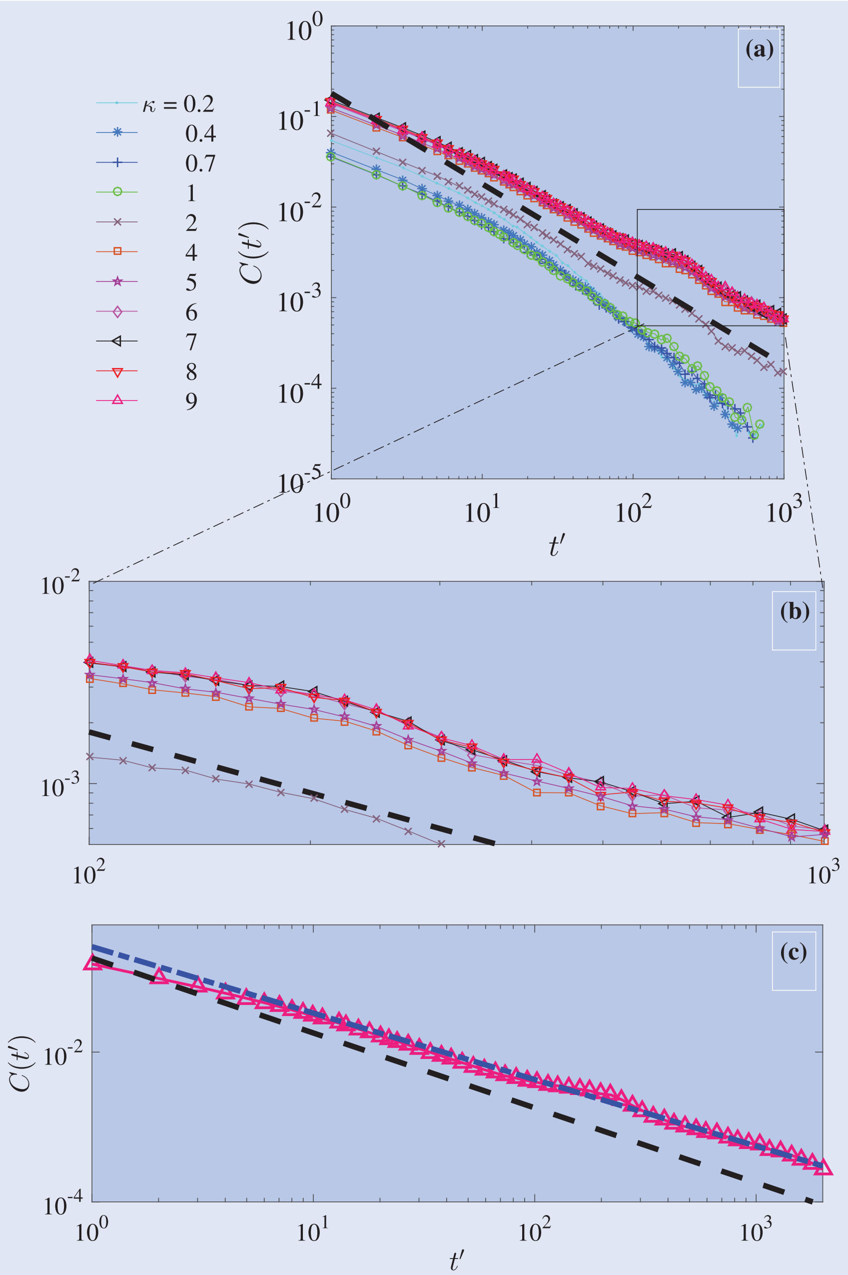

Figure 3. (a) The ACF of the binary sequence of defaults corresponding to 11 different values of κ. The dashed black line corresponds to a power-law with exponent –1, which is the largest exponent that leads to clustering (see remark 1). (b) A blow-up of the rectangular area shown in panel (a) illustrating the coalescence of $C(t')$ for large values of the degree of heterogeneity, $\kappa = \{6, 7, 8, 9\}$. (c) The ACF corresponding to $\kappa = 9$, averaged over 5×10^2 different realisations of s_j (red upright triangles). The blue dot-dashed line is the result of fitting $C(t')$ with a power-law model $C(t') \propto t'^{-\eta}$, $\eta = 0.887 \pm 0.003$ ($R^2 = 0.9927$). The power-law with exponent -1 is also shown for the sake of comparison (black dashed line).

limiting form exhibiting a power-law trend with an exponent less than 1 (in absolute value).

To gain some insight into the qualitative difference with respect to the persistence of correlations between defaults as a function of the degree of heterogeneity κ, let us turn our attention to the default statistics. In figure 4, we present the aggregate PDF of waiting times between defaults† using a logarithmic scale on both axes for 6 different values of κ. We observe that for small degrees of heterogeneity $\kappa = \{0.2, 0.4, 0.7\}$ the density function asymptotically decays approximately exponentially. This is better demonstrated in the inset where we use semi-logarithmic axes.‡ On the contrary, for sufficiently large heterogeneity—such that the corresponding ACFs have converged to the limiting form—the PDFs exhibit a constant decay rate in the doubly logarithmic plot (power-law tail). Fitting the aggregate density for $\kappa = 9$,§ corresponding to the

† The PDF of waiting-times between default is also known as the failure function in survival analysis theory.

‡ The use of a logarithmic scale for the vertical axis transforms an exponential function to a linear one.

§ To increase the accuracy of the fit, we increase the number of realisations of s_j to 10^3.

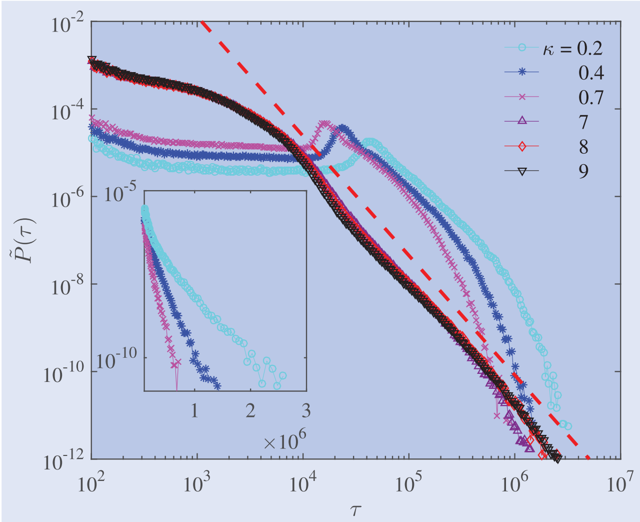

Figure 4. The aggregate PDF of waiting times between defaults for six different degrees of heterogeneity using double logarithmic scale. For large heterogeneity $\kappa = \{7, 8, 9\}$, we observe that the PDF is decaying approximately linearly, corresponding to a power-law decay. Performing a fit with the model $\tilde{P}(\tau) \sim \tau^{-\zeta}$ we obtain $\zeta = 2.84 \pm 0.03$ ($R^2 = 0.9947$). To illustrate the approximate exponential asymptotic decay of the aggregate PDF for $\kappa = \{0.2, 0.4, 0.7\}$ we also show the corresponding aggregate densities using a logarithmic scale on the vertical axis (inset).

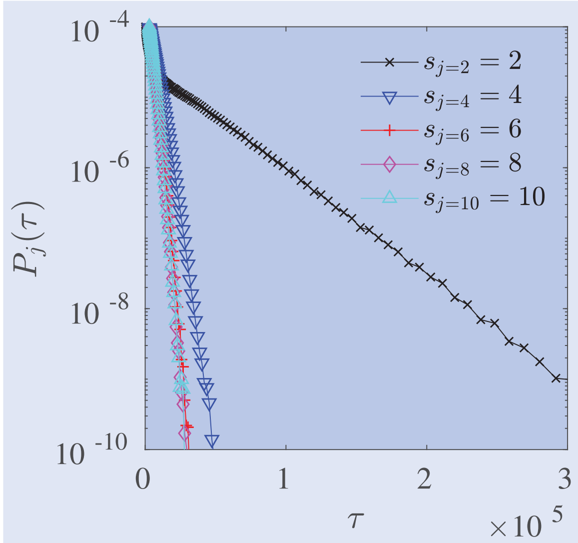

Figure 5. The PDF of waiting times between defaults τ for specific HFs, having different responsiveness $s_j = \{2, 4, 6, 8, 10\}$ (black diagonal crosses, downright triangles, red upright crosses, magenta diamonds and cyan upright triangles, respectively). Note the log-linear scale.

highest degree of heterogeneity considered, with the model $\tilde{P}(\tau) \sim \tau^{-\zeta}$ we obtain $\zeta = 2.84 \pm 0.03$ (red dashed line).

Default statistics on an individual level. Let us now turn our attention to the statistical properties of HFs on a microscopic scale, i.e. study each HF default statistics individually. In figure 5, we show as an example the density function $P_j(\tau)$, of waiting times τ between defaults, for a number of HFs corresponding to high heterogeneity, $\kappa = 9$, with $s_j = \{2, 4, 6, 8, 10\}$ on a log-linear scale. The results were obtained by iterating the model for 3×10^8 periods and averaging over 100 different initial conditions,† holding s_j fixed at $\{1, 2, \ldots 10\}$. We observe that $P_j(\tau)$ for $\tau \gg 1$ decays linearly, and thus it can be well described by an exponential function. Consequently, all HFs on a microscopic level—individually—are characterised by exponential PDFs of waiting-times, and therefore the default events approximately follow a Poisson process. The stability of each HF, quantified by the probability of default per time-step μ_j, is different for each HF, and depends on its responsiveness s_j. This is reflected by the different slopes of the approximately straight lines shown in figure 5 for the different values of s_j.

Thus the default statistics on an aggregate level are qualitatively different for large values of κ compared to the corresponding ones observed when each HF is studied individually. Moreover, we have already established that for such high values of the degree of heterogeneity the defaults are clustered. In the following, we will investigate how the emergence of a fat-tail in the aggregate statistics is connected with the observed clustering of defaults.

† We are averaging using different seeds for the pseudo-random number generator used in equation (4).

4.2. Analytical results

The goal of this section is to show that when the default statistics of HFs are individually described by (different) Poisson processes (due to the heterogeneity among the HFs), we obtain a qualitatively different result after aggregation: the aggregate PDF of the waiting-times between defaults exhibits a power-law tail for long waiting-times. Also, if the relative proportion of very stable HFs approaches 0 sufficiently slowly, then defaults will form clusters.

We start by utilising a key observation arising from our numerical results, namely the observation that $P_j(\tau)$, for $\tau \gg 1$ decays linearly (in log-linear scale) and thus it can be well described by an exponential function. Therefore we can assume that

$$P_j(\tau; \tau \gg 1) \sim \mu_j \exp(-\mu_j \tau), \quad \forall j \in \{1, \ldots, 10\}. \tag{9}$$

Conditional on the above, we know that for sufficiently long waiting times between defaults, default events of individual HFs have the following statistical properties: (i) they are approximately independent and (ii) occur with a well-defined mean probability per unit time step. From this we derive that the probability $P_j(T = \tau)$, $\tau \in \mathbb{N}_+$, is given by a geometric probability mass function (PMF)

$$P_j(\tau) = p_j(1 - p_j)^{\tau - 1}, \tag{10}$$

where p_j denotes the probability of default of the jth HF.

Given that our focus is in the asymptotic properties of the PDFs, T can be treated as a continuous variable. In this limit, the renewal process given in equation (10) becomes a Poisson process; and the geometric PMF tends to an exponential PDF.† Thus equation (9) can be approximated by equation (10).

The question then arises as to how the aggregation of these very simple stochastic processes can lead to the non-trivial fat-tailed statistics we observed in figure 4 for a sufficiently high degree of heterogeneity. Note that the aggregate PDF $\tilde{P}(\tau)$ we seek to obtain is a result of the mixing of the Poisson processes governing each of the HFs. In the limit of a continuum of HFs the aggregate distribution is

$$\tilde{P}(\tau) = \int_0^\infty \mu \exp(-\mu\tau)\rho(\mu)\,\mathrm{d}\mu, \tag{11}$$

where $\rho(\mu)$ stands for the PDF of μ given the responsiveness s_j.‡

Assumption 1 $\rho(\mu)$ in a neighbourhood of 0 can be expanded in a power series of the form $\rho(\mu) = \mu^\nu \sum_{k=0}^n c_k\mu^k + R_{n+1}(\mu)$, with $\nu > -1$.§

This assumption is quite general and only excludes functions that behave pathologically in a neighbourhood around 0. Then from equation (9) and assumption 1 we can show that the aggregation of the exponential densities determining the default statistic for each HF individually leads to a qualitatively different heavy-tailed PDF.

Let $\mu^j \in \mathbb{R}^+$ be the mean default rate of the jth HF, contributing at the aggregate level with a statistical weight $\rho(\mu)$, which is determined by the interactions between the agents in the market and the distribution of the responsiveness s.

Proposition 1 *Consider the exponential density function $P(\tau;\mu)$ describing the individual default statistics of an HF. It follows then from assumption 1, that the aggregate PDF $\tilde{P}(\tau)$ exhibits a power-law tail.*

Proof The aggregate density can be viewed as the Laplace transform $\mathcal{L}[.]$ of the function $\phi(\mu) \equiv \mu\rho(\mu)$, with respect to μ. Hence,

$$\tilde{P}(T=\tau) \equiv \mathcal{L}[\phi(\mu)](\tau) = \int_0^\infty \phi(\mu)\exp(-\mu\tau)\,\mathrm{d}\mu. \tag{12}$$

To complete the proof we apply Watson's Lemma (Debnath and Bhatta 2007, p. 171) to the function $\phi(\mu)$, according to which the asymptotic expansion of the Laplace transform of a function $f(\mu)$ that admits a power-series expansion in a neighbourhood of 0 (see assumption 1) of the form $f(\mu) = \mu^\nu \sum_{k=0}^n b_k\mu^k + R_{n+1}(\mu)$, with $\nu > -1$ is

$$\mathcal{L}_\mu[f(\mu)](\tau) \sim \sum_{k=0}^n b_k \frac{\Gamma(\nu+k+1)}{\tau^{\nu+k+1}} + O\left(\frac{1}{\tau^{\nu+n+2}}\right). \tag{13}$$

Given that $\phi(\mu)$ for $\mu \to 0_+$ is

$$\phi(\mu) = \mu^{\nu+1}\sum_{k=0}^n c_k\mu^k + R_{n+1}(\mu), \tag{14}$$

we conclude that

$$\tilde{P}(\tau) \propto \frac{1}{\tau^{k+\nu+2}} + O\left(\frac{1}{\tau^{k+\nu+3}}\right). \tag{15}$$

■

Corollary 1 *If $0 < k+\nu \le 1$, then the variance of the aggregate density diverges (shows a fat tail). However, the expected value of τ remains finite.*

An important aspect of the emergent heavy-tailed statistics stemming from the heterogeneous behaviour of the HFs, is the absence of a characteristic time-scale for the occurrence of defaults (scale-free asymptotic behaviour¶). Thus even if each HF defaults according to a Poisson process with intensity $\mu(s)$—which has the intrinsic characteristic time-scale $1/\mu(s)$—after aggregation this property is lost due to the mixing of all the individual time-scales. Therefore, on a macroscopic level, there is no characteristic time-scale, and all time-scales, short and long, become relevant.

This characteristic becomes even more prominent if the density function $\rho(\mu)$ is such that the resulting aggregate density becomes fat-tailed, i.e. the variance of the aggregate distribution diverges. In this case, extreme values of waiting times between defaults will be occasionally observed, deviating far from the mean. This will leave a particular 'geometrical' imprint on the sequence of default times. Defaults occurring close together in time (short waiting times τ) will be clearly separated due to the non-negligible probability assigned to long waiting times. Consequently, defaults, macroscopically, will have a 'bursty' or intermittent, character, with long quiescent periods of time without the occurrence of defaults and 'violent' periods during which many defaults are observed close together in time. Hence, infinite variance of the aggregate density will result in the clustering of defaults.

In order to show this analytically, we construct a binary sequence by mapping time-steps when no default events occur to 0 and 1 otherwise. As we show below, if the variance of the aggregate distribution is infinite, then the autocorrelation function of the binary sequence generated in this manner, exhibits a power-law asymptotic behaviour with an exponent $\beta < 1$. Therefore, the ACF is non-summable and consequently, according to definition 1 defaults are clustered.

Let T_i, $i \in \mathbb{N}_+$, be a sequence of times when one or more HFs default and assume that the PDF of waiting times between defaults $\tilde{P}(\tau)$, for $\tau \to \infty$, behaves (to leading order) as $\tilde{P}(\tau) \propto \tau^{-a}$. Consider now the renewal process

† This limit is valid for $\tau \gg 1$ and $p_j \ll 1$ such that $\tau p_j = \mu_j$, where μ_j is the parameter of the exponential PDF (see equation 9) (Nelson 1995).

‡ The distribution function of the random parameter μ is also known as the *structure* or *mixing* distribution (Beichelt 2006).

§ Since $\rho(\mu)$ is a PDF it must be normalisable and thus, a singularity at $\mu = 0$ must be integrable.

¶ If a function $f(x)$ is a power-law, i.e. $f(x) = cx^a$, then a rescaling of the independent variable of the form $x \to bx$ leaves the functional form invariant ($f(x)$ remains a power-law). In fact, a power-law functional form is a necessary and sufficient condition for scale invariance (Farmer and Geanakoplos 2008). This scale-free behaviour of power-laws is intimately linked with concepts such as self-similarity and fractals (Mandelbrot 1983).

$S_m = \sum_{i=0}^{m} T_i$. Let $Y(t) = 1_{[0,t]}(S_m)$, where $1_A : \mathbb{R} \to \{0,1\}$ denotes the indicator function, satisfying

$$1_A = \begin{cases} 1 & : \quad x \in A \\ 0 & : \quad x \notin A \end{cases}$$

THEOREM 1 *If the variance of the density function* $\tilde{P}(\tau)$ *diverges, i.e.* $2 < a \leq 3$, *then the ACF of* $Y(t)$,

$$C(t') = \frac{\mathbb{E}\left[Y_{t_0} Y_{t_0+t'}\right] - \mathbb{E}\left[Y_{t_0}\right]\mathbb{E}\left[Y_{t_0+t'}\right]}{\sigma_Y^2},$$

where t_0, $t' \in \mathbb{R}$ *and* σ_Y^2 *is the variance of* $Y(t)$, *for* $t \to \infty$ *decays as*

$$C(t') \propto t'^{2-\alpha} \tag{16}$$

The proof can be found in Appendix 2.

Turning back to the numerical results shown in figure 4, the aggregate PDF as already discussed converges to a limiting form, characterised by a fat-tail with an exponent equal -2.84 ± 0.03. Therefore, from equation (16) we deduce that the ACF should show a power-law trend with exponent -0.84 ± 0.03. The result of the regression of the ACF for $\kappa = 9$ was -0.887 ± 0.003 (blue dashed-dotted line in figure 3 c), in good agreement with the analytical result.

In this section, we have shown that when the default statistics of HFs are individually described by (different) Poisson processes (due to the heterogeneity in the prior beliefs among the HFs) we obtain a qualitatively different result after aggregation: the aggregate PDF of the waiting-times between defaults exhibits a power-law tail for long waiting-times. As shown in proposition 1, if the relative proportion of very stable HFs approaches 0 sufficiently slowly (at most linearly with respect to the individual default rate μ, as $\mu \to 0$), then long waiting-times between defaults become probable, and as a result, defaults which occur closely in time will be separated by long quiescent time periods and defaults will form clusters. The latter is quantified by the non-integrability of the ACF of the sequence of default times, signifying infinite memory of the underlying stochastic process describing defaults on the aggregate level. It is worth commenting on the fact that the most stable (in terms of defaults) HFs are responsible for the appearance of a fat-tail in the aggregate PDF.

4.3. Link with empirical evidence

A clear message of the paper is that when the degree of heterogeneity is sufficiently large, the default of HFs would be clustered. The clustering of default is supported by the empirical findings of Boyson *et al.* (2010), who find a positive correlation between the default of hedge funds upon and beyond the one related to the autocorrelation of stock returns.

Also, our study sheds light on why and when clustering in the defaults would arise. Therefore, our work opens the doors to future empirical work that revisits the findings of Boyson *et al.* (2010), in an attempt to verify the channels driving the positive correlation between the default of HFs. In addition, an empirical prediction of our study is that the clustering of default is more likely for HFs that invest in complex assets. The underlying rationale is that for complex assets, there is limited consensus among financial analysts about the assetâs fundamental value, or equivalently, the degree of heterogeneity is larger.

5. Concluding remarks

This paper studied the role of the heterogeneity in available information among different HFs in the emergence of clustering of defaults. The economic mechanism leading to the clustering of defaults is related to the leverage cycle put forward by Geanakoplos and coauthors. In these models, the presence of leverage in a market leads to the overpricing of the collateral used to back-up loans during a boom, whereas during a recession, collateral becomes depreciated due to a synchronous de-leveraging compelled by the creditors. In the present work, we have shown that this feedback effect between market volatility and margin requirements is a necessary, yet not a sufficient condition for the clustering of defaults and, in this sense, the emergence of systemic risk.

We have shown that a large difference in the expectations of the HFs is an essential ingredient for defaults to be clustered. We show that when the degree of heterogeneity (realised in our model in terms of the beliefs across HFs about the volatility of the market) is sufficiently high, poorly performing HFs are able to absorb shocks caused by fire-sales. As a result, they obtain a larger than usual market share and improve their performance. In this fashion, a default due to their poor performance is delayed, allowing them to remain in operation until the downturn of the next leverage cycle. This leads to an increase in the probability of poorly and high-performing hedge funds to default in sync at a later time, and thus the probability of collective defaults.

This manifests itself in the emergence of heavy-tailed (scale-free) statistics on the aggregate level. We show that this scale-free character of the aggregate survival statistics, when combined with large fluctuations of the observed waiting-times between defaults, i.e. infinite variance of the corresponding aggregate PDF, leads to the presence of infinite memory in the default time sequence. Consequently, the probability of observing a default of an HF in the future is much higher if one (or more) is observed in the past, and as such, defaults are clustered.

Interestingly, a slow-decaying PDF of waiting-times, which inherently signifies a non-negligible measure of extremely stable HFs, is shown to be directly connected with the presence of infinite memory. Therefore, our work shows that individual stability can lead to market-wide risk.

The leverage cycle theory correctly emphasises the importance of collateral, in contrast to the conventional view, according to which the interest rate completely determines the demand and supply of credit. However, the feedback loop created by the volatility of asset prices and margin constraints poses a systemic risk only if the market is sufficiently heterogeneous so that ‘pessimistic’ players, who individually are very stable, exceed a critical mass.

This work raises several interesting questions, which we aim to address in the future. In this paper, we have assumed that the difference in beliefs is due to disagreement about the long-run volatility of the risky asset, and remains constant over time, i.e. the agents do not update their beliefs given their observations. This assumption is crucial in order to be able to analyse the effects of different degrees of heterogeneity. Regarding this issue, future work can take two different directions: on the one hand, this assumption can be relaxed, allowing agents to update their beliefs on market volatility. However, given that market volatility is endogenous, it is not guaranteed that agents' beliefs will converge. On the other hand, we can study the effects of heterogeneity stemming from different aversion to risk among the HFs, while retaining the common prior assumption. Furthermore, these two approaches can be combined by assuming both different aversion to risk, and different beliefs about price volatility. Finally, our work can also be extended in two further directions. The first being to give a more active role to the bank which provides the loans, while the second is to study the effects of different regulations on credit supply.

Disclosure statement

No potential conflict of interest was reported by the authors.

ORCID

S. Terovitis http://orcid.org/0000-0001-9529-0386

References

Acharya, V.V. and Viswanathan, S., Leverage, moral hazard, and liquidity. *J. Finance*, 2011, **66**, 99–138.
Adrian, T. and Shin, H.S., Liquidity and leverage. *J. Financ. Intermed.*, 2010, **19**, 418–437.
Ait-Sahalia, Y., Parker, J.A. and Yogo, M., Luxury goods and the equity premium. *J. Finance.*, 2004, **59**, 2959–3004.
Andersen, T.G. and Bollerslev, T., Heterogeneous information arrivals and return volatility dynamics: Uncovering the long-run in high frequency returns. *J. Finance.*, 1997, **52**, 975–1005.
Ang, A., Gorovyy, S. and van Inwegen, G.B., Hedge fund leverage. *J. Financ. Econ.*, 2011, **102**, 102–126.
Baillie, R.T., Long memory processes and fractional integration in econometrics. *J. Econom.*, 1996, **73**, 5–59.
Beichelt, F., *Stochastic Processes in Science, Engineering and Finance*, 2006 (Chapman & Hall/CRC: Boca Raton).
Boyson, N.M., Stahel, C.W. and Stulz, R.M., Hedge fund contagion and liquidity shocks. *J. Finance*, 2010, **65**, 1789–1816.
Brock, W.A. and Hommes, C.H., A rational route to randomness. *Econometrica*, 1997, **65**, 1059–1096.
Brock, W.A. and Hommes, C.H., Heterogeneous beliefs and routes to chaos in a simple asset pricing model. *J. Econom. Dyn. Control*, 1998, **22**, 1235–1274.
Brock, W.A. and LeBaron, B.D., A dynamic structural model for stock return volatility and trading volume. *Rev. Econ. Stat.*, 1996, **78**, 94–110.
Brunnermeier, M.K. and Pedersen, L.H., Market liquidity and funding liquidity. *Rev. Financ. Stud. Soc. Financ. Stud.*, 2009, **22**, 2201–2238.
Brunnermeier, M.K. and Sannikov, Y., A macroeconomic model with a financial sector. *Am. Econom. Rev.*, 2014, **104**, 379–421.
Campbell, J.Y. and Viceira, L.M., *Strategic Asset Allocation: Portfolio Choice for Long-term Investors*, 2002 (Oxford University Press: New York).
Chiarella, C., The dynamics of speculative behaviour. *Operations Ann. Oper. Res.*, 1992, **37**, 101–123.
Chiarella, C. and Di Guilmi, C., The financial instability hypothesis: A stochastic microfoundation framework. *J. Econom. Dyn. Control*, 2011, **35**, 1151–1171.
Chiarella, C. and He, X.Z., Heterogeneous beliefs, risk and learning in a simple asset pricing model. *Comput. Econom.*, 2002, **19**, 95–132.
Chiarella, C., Dieci, R. and He, X.Z., Heterogeneity, market mechanisms, and asset price dynamics. In *Handbook of Financial Markets: Dynamics and Evolution*, edited by T. Hens and K.R. Schenk-Hoppe, pp. 277–344, 2009 (Elsevier),
Chiarella, C., He, X.Z. and Zwinkels, R.C., Heterogeneous expectations in asset pricing: Empirical evidence from the S&P500. *J. Econ. Behav. Organ.*, 2014, **105**, 1–16.
Danthine, J.-P. and Donaldson, J., *Intermediate Financial Theory*, 2005 (Academic Press: London).
Day, R.H. and Huang, W., Bulls, bears and market sheep. *J. Econ. Behav. Organ.*, 1990, **14**, 299–329.
Debnath, L. and Bhatta, D., *Integral Transforms and Their Applications*, 2nd ed., 2007 (Chapman & Hall/CRC: Boca Raton).
Di Guilmi, C. and Carvalho, L., The dynamics of leverage in a demand-driven model with heterogeneous firms. *J. Econ. Behav. Organ.*, 2017, **140**, 70–90.
Di Guilmi, C., He, X.Z. and Li, K., Herding, trend chasing and market volatility. *J. Econom. Dyn. Control*, 2014, **48**, 349–373.
Farmer, J.D. and Geanakoplos, J., Power laws in economics and elsewhere, 2008.
Fostel, A. and Geanakoplos, J., Leverage cycles and the anxious economy. *Am. Econom. Rev.*, 2008, **98**, 1211–1244.
Fostel, A. and Geanakoplos, J., Tranching, CDS, and asset prices: How financial innovation can cause bubbles and crashes. *Am. Econom. J. Macroeconom.*, 2012, **4**, 190–225.
Fostel, A. and Geanakoplos, J., Endogenous collateral constraints and the leverage cycle. *Annu. Rev. Econom.*, 2014, **6**, 771–799.
Fostel, A. and Geanakoplos, J.D., Leverage and default in binomial economies: A complete characterization. *Econometrica*, 2015, **83**, 2191–2229.
Fostel, A. and Geanakoplos, J., Financial innovation, collateral, and investment. *Am. Econom. J. Macroeconom.*, 2016, **8**, 242–84.
Geanakoplos, J., *Promises, Promises*, Vol. 28., 1997 (Addison-Wesley Reading: MA).
Geanakoplos, J., Liquidity, default, and crashes endogenous contracts in general. In *Proceedings of the Advances in Economics and Econometrics: Theory and Applications: Eighth World Congress*, Vol. 170, 2003.
Geanakoplos, J., The leverage cycle. In *NBER Macroeconomics Annual 2009, Volume 24*, pp. 1–65, 2010 (National Bureau of Economic Research, Inc),
Geanakoplos, J., Leverage, default, and forgiveness: Lessons from the American and European crises. *J. Macroecon.*, 2014, **39**, 313–333.
Geanakoplos, J. and Zame, W., Collateral and the enforcement of intertemporal contracts. Cowles foundation discussion papers, Cowles Foundation for Research in Economics, Yale University, 1997.
Geanakoplos, J. and Zame, W., Collateral equilibrium, I: A basic framework. *Econom. Theory*, 2014, **56**, 443–492.
Gregoriou, G.N., Buebner, G., Papageorgiou, N. and Rouah, F., *Hedge Funds: Insights in Performance Measurement, Risk Analysis, and Portfolio Allocation*, 2005 (John Wiley & Sons, Inc.: Hoboken).
Gromb, D. and Vayanos, D., Equilibrium and welfare in markets with financially constrained arbitrageurs. *J. Financ. Econ.*, 2002, **66**, 361–407.
Haghani, S., Modeling hedge fund lifetimes: A dependent competing risks framework with latent exit types. *J. Empir. Financ.*, 2014, **28**, 291–320.

He, X.Z. and Li, Y., Power-law behaviour, heterogeneity, and trend chasing. *J. Econom. Dyn. Control*, 2007, **31**, 3396–3426.
He, X.Z., Li, K. and Shi, L., Social Interaction, Stochastic Volatility, and Momentum. Mimeo, 2017.
Hommes, C.H., Chapter 23 heterogeneous agent models in economics and finance. In *Handbook of Computational Economics*, Vol. 2, edited by L. Tesfatsion and K. Judd, pp. 1109–1186, 2006 (Elsevier),
Iori, G., A microsimulation of traders activity in the stock market: The role of heterogeneity, agents' interactions and trade frictions. *J. Econ. Behav. Organ.*, 2002, **49**, 269–285.
Jeffrey, A. and Zwillinger, D., *Table of Integrals, Series, and Products*, Table of Integrals, Series, and Products Series, 2007 (Elsevier Science).
Kodres, L.E. and Pritsker, M., A rational expectations model of financial contagion. *J. Finance*, 2002, **57**, 769–799.
Kyle, A.S. and Xiong, W., Contagion as a wealth effect. *J. Finance*, 2001, **56**, 1401–1440.
LeBaron, B., Agent-based computational finance. In *Handbook of Computational Economics*, edited by L. Tesfatsion and K.L. Judd, Vol. 2, chap. 24, pp. 1187–1233, 2006 (Elsevier),
Levy, M., Stock market crashes as social phase transitions. *J. Econom. Dyn. Control*, 2008, **32**, 137–155.
Lux, T., Herd behaviour, bubbles and crashes. *Econom. J.*, 1995, **105**, 881–96.
Lux, T., The socio-economic dynamics of speculative markets: Interacting agents, chaos, and the fat tails of return distributions. *J. Econ. Behav. Organ.*, 1998, **33**, 143–165.
Lux, T. and Marchesi, M., Scaling and criticality in a stochastic multi-agent model of a financial market. Discussion Paper Series B 438, 1999 (University of Bonn).
Mandelbrot, B.B., *The Fractal Geometry of Nature*, 1983 (W. H. Freeman and Company: New York).
Nelson, H., *Probability, Stochastic Processes, and Queueing Theory: The Mathematics of Computer Performance Modeling*, 1995 (Springer: New York).
Poledna, S., Thurner, S., Farmer, J.D. and Geanakoplos, J., Leverage-induced systemic risk under Basle II and other credit risk policies. *J. Bank. Financ.*, 2014, **42**, 199–212.
Procaccia, I. and Schuster, H.G., Functional renormalization-group theory of universal 1/f noise in dynamical systems. *Phys. Rev. A*, 1983, **28**, 1210.
Ryoo, S., Long waves and short cycles in a model of endogenous financial fragility. *J. Econ. Behav. Organ.*, 2010, **74**, 163–186.
Samorodnitsky, G., Long range dependence. *Found. Trends. Stoch. Sys.*, 2007, **1**, 163–257.
Simsek, A., Belief disagreements and collateral constraints. *Econometrica*, 2013a, **81**, 1–53.
Simsek, A., Speculation and risk sharing with new financial assets. *Q. J. Econ.*, 2013b, **128**, 1365–1396.
Statista, Assets under management of hedge funds worldwide 1997–2018. Accessed: 2019-4-01.
Stowell, D.P., *An Introduction to Investment Banks, Hedge Funds, and Private Equity: The New Paradigm*, 2010 (Elsevier Inc.: San Diego).
Szpiro, G.G., Relative risk aversion around the world. *Econ. Lett.*, 1986, **20**, 19–21.
Szpiro, G.G. and Outreville, J.F., Relative risk aversion around the world: Further results. *J. Bank. Financ.*, 1988, **6**, 127–128.
Thurner, S., Farmer, J.D. and Geanakoplos, J., Leverage causes fat tails and clustered volatility. *Quant. Finance*, 2012, **12**, 695–707.
Vayanos, D. and Wang, J., Liquidity and asset returns under asymmetric information and imperfect competition. *Rev. Financ. Stud.*, 2012, **25**, 1339–1365.
Xiong, W., Convergence trading with wealth effects: An amplification mechanism in financial markets. *J. Financ. Econ.*, 2001, **62**, 247–292.

Appendices

Appendix 1. Optimal Demand

We seek to determine the optimal demand for each of the HFs given their beliefs about price volatility I^j. This translates into the optimisation problem, assuming log-normal returns on the risky asset

$$\underset{D_t^j\in[0,D_{t,\max}]}{\operatorname{argmax}} \left\{\mathbb{E}\left[U(\gamma W_{t+1}^j)|I^j\right]\right\}, \tag{A1}$$

where $U(\gamma W_{t+1}^j)=\gamma {W_{t+1}^j}^{1-a}/(1-a)\sim {W_{t+1}^j}^{1-a}$, and W_{t+1}^j is the wealth of the jth HF at the next period. To simplify the notation, in the following we will assume that the expected value, and variance are always conditioned on HF's prior beliefs, and moreover, we will drop the superscript j. Equation (A1) is equivalent to the maximisation of the logarithm of the expected utility. Furthermore, given that returns are log-normally distributed, it follows that Campbell and Viceira (2002, pp. 17-21)

$$\log \mathbb{E}\left[{W_{t+1}}^{1-a}\right]=\mathbb{E}\left[\log {W_{t+1}}^{1-a}\right]+\frac{\operatorname{Var}\left[\log {W_{t+1}}^{1-a}\right]}{2} \tag{A2}$$

Consequently, the problem becomes

$$\underset{D_t\in[0,D_{t,\max}]}{\operatorname{argmax}} \left\{(1-a)\,\mathbb{E}\left[\log W_{t+1}\right] + (1-a)^2\frac{\operatorname{Var}\left[\log W_{t+1}\right]}{2}\right\}. \tag{A3}$$

The wealth of the *jth* HF at the next period is

$$W_{t+1}=(1-\gamma)(1+x_tR_{t+1})W_t, \tag{A4}$$

where x is the fraction of its wealth invested into the risky asset, and R the (arithmetic) return of the portfolio. Re-expressing equation (A4) in terms of the logarithmic returns r we get

$$\log\left(W_{t+1}\right)=\log W_t+\log\left[1+x_t\left(\exp(r_{t+1})-1\right)\right]+\log(1-\gamma), \tag{A5}$$

albeit a transcendental equation with respect to r. An approximative solution can be obtained by performing a Taylor expansion of equation (A5) with respect to r to obtain

$$\log(W_{t+1})=\log(W_t)+x_tr_{t+1}\left(1+\frac{r_{t+1}}{2}\right) - \frac{{x_t}^2}{2}r_{t+1}^2+\log(1-\gamma)+\mathcal{O}\left(r^3\right). \tag{A6}$$

Substituting equation (A6) into equation (A3), and furthermore approximating $\mathbb{E}(r_{t+1}^2)$ with $\operatorname{Var}(r_{t+1})$ we obtain

$$\underset{D_t\in[0,\lambda_{\max}]}{\operatorname{argmax}} \left\{\log W_t+x_t\mathbb{E}(r_{t+1})+\frac{x_t}{2}(1-x_t)\operatorname{Var}(r_{t+1}) + \log(1-\gamma)+(1-a)x_t^2\operatorname{Var}(r_{t+1})\right\}. \tag{A7}$$

Finally the first-order condition yields

$$x_t=\min\left[\frac{\mathbb{E}(r_{t+1})+\frac{1}{2}\operatorname{Var}(r_{t+1})}{a\operatorname{Var}(r_{t+1})},\lambda_{\max}\right]. \tag{A8}$$

Consequently, ignoring autocorrelation in prices, the optimal demand for HF j in terms of the number of shares of the risky asset given the price at the current period is

$$D_t=\min\left\{\frac{\log(V/p_t)+\frac{1}{2}\operatorname{Var}\left[\log p_{t+1}|I^j\right]}{a\operatorname{Var}\left[\log p_{t+1}|I^j\right]},\lambda_{\max}\right\}\frac{W_t}{p_t}. \tag{A9}$$

Appendix 2. Proof of theorem 1

As already stated in section 4.2, theorem 1, assuming that the process defined by $Y(t) = 1_{[0,t]}(S_m)$ is ergodic, the auto-correlation function can be expressed as a time-average

$$C(t') \propto \lim_{K\to\infty} \frac{1}{K} \sum_{t=0}^{K} Y_t Y_{t+t'}. \tag{A10}$$

Given that $Y(t)$ is by definition a binary variable, the only non-zero terms contributing to the sum appearing on the right-hand side (RHS) of equation (A10) correspond to default events (mapped to 1) that occur with a time difference equal to t'. Therefore, the RHS of equation (A10) is proportional to the conditional probability of observing a default at time t', given that a default has occurred at time $t = 0$. Therefore, we can express $C(t')$ in terms of the aggregate probability $\tilde{P}(\tau = t')$, i.e. the probability of a default event being observed after t' time-steps, given that one has just been observed. Moreover, we must take into account all possible combinations of defaults happening at times $t < t'$. For example, let us assume that we want to calculate $C(t' = 2)$. In this case, there are exactly two possible set of events that would give a non-zero contribution. Either a default happening exactly two time-steps after the last one (at $t = 0$), or two subsequent defaults happening at $t = 1$, and $t = 2$. In this fashion, we can express the correlation function in terms of the probability the waiting-times between defaults as Procaccia and Schuster (1983),

$$C(1) = \tilde{P}(1), \tag{A11}$$

$$\begin{aligned} C(2) &= \tilde{P}(2) + \tilde{P}(1)\tilde{P}(1) \\ &= \tilde{P}(2) + \tilde{P}(1)C(1), \end{aligned} \tag{A12}$$

$$\vdots$$

$$C(t') = \tilde{P}(t') + \tilde{P}(t'-1)C(1) + \ldots \tilde{P}(1)C(t'-1). \tag{A13}$$

If we further define $C(0) = 1$ and $\tilde{P}(0) = 0$, then equation (A13) can be written more compactly as

$$C(t') = \sum_{\tau=0}^{t'} C(t'-\tau)\tilde{P}(\tau) + \delta_{t',0}, \tag{A14}$$

where $\delta_{t',0}$ is the Kronecker delta.

We are interested only in the long time limit of the ACF. Hence, we can treat time as a continuous variable and solve equation (A14) by applying the Laplace transform $\mathcal{L}\{f(\tau)\}(s) = \int_0^\infty f(\tau)\exp(-s\tau)d\tau$, utilising also the convolution theorem.Taking these steps, we obtain

$$C(s) = \frac{1}{1 - \tilde{P}(s)}, \tag{A15}$$

where $\tilde{P}(s) = \mathcal{L}\{\tilde{P}(\tau)\}(s) = \int_0^\infty \tilde{P}(\tau)\exp(-s\tau)d\tau$. We will assume that $\tilde{P}(\tau) \propto \tau^{-a}$ for any $\tau \in [1, \infty)$, i.e. the asymptotic power-law behaviour ($\tau \gg 1$) will be assumed to remain accurate for all values of τ. Under this assumption,

$$\tilde{P}(\tau) = \begin{cases} A\tau^{-a}, & \tau \in [1, \infty), \\ 0, & \tau \in [0, 1). \end{cases}, \tag{A16}$$

where $A = 1/\int_1^\infty \tau^{-a}\,\mathrm{d}\tau = a - 1$. The Laplace transform of equation (A16) is,

$$\tilde{P}(s) = (a-1)E_a(s), \tag{A17}$$

where $E_a(s)$ denotes the exponential integral function defined as

$$E_a(s) = \int_1^\infty \exp(-st)\, t^{-a}\mathrm{d}t/;\ \mathrm{Re}(s) > 0. \tag{A18}$$

The inversion of the Laplace transform after the substitution of equation (A17) in equation (A15) is not possible analytically. However, we can easily derive the correlation function in the Fourier space (known as the power spectral density function) $\mathcal{F}\{C(t')\}(f) = \sqrt{\frac{2}{\pi}}\int_0^\infty C(t')\cos(2\pi f)dt'$ by the use of the identity (Jeffrey and Zwillinger 2007, p. 1129),

$$\mathcal{F}\{C(t')\} = \frac{1}{\sqrt{2\pi}}\left[C(s \to 2\pi i f) + C(s \to -2\pi i f)\right], \tag{A19}$$

relating the Fourier cosine transform $\mathcal{F}\{g(t)\}(f)$, of a function $g(t)$, to its Laplace transform $g(s)$, to obtain,

$$C(f) = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{1-(a-1)E_a(2if\pi)} + \frac{1}{1-(a-1)E_a(-2if\pi)}\right) \tag{A20}$$

From equation (A20), we can readily see that as $f \to 0_+$ (equivalently $t' \to \infty$), $C(f) \to \infty$. To derive the asymptotic behaviour of $C(f)$ we expand about $f \to 0_+$ (up to linear order) using

$$E_a(2if\pi) = ai^{a+1}(2\pi)^{a-1}f^{a-1}\Gamma(-a) - \frac{2i\pi f}{a-2} + \frac{1}{a-1} + O(f^2) \tag{A21}$$

to obtain

$$\begin{aligned} &C(f) \\ &= -\frac{i\sqrt{2\pi}(a-2)f}{4\pi^2(a-1)f^2 + \left(2^{a+1}\pi^a(if)^a - a(2i\pi)^a f^a\right)\Gamma(2-a)} \\ &\quad + \frac{i\sqrt{2\pi}(a-2)f}{4\pi^2(a-1)f^2 + \left(2^{a+1}\pi^a(-if)^a - a(-2i\pi)^a f^a\right)\Gamma(2-a)}. \end{aligned} \tag{A22}$$

After some algebraic manipulation, for $f \to 0$ equation (A22) yields

$$C(f) = Af^{a-3}, \tag{A23}$$

where

$$A = -\frac{2^{a+\frac{1}{2}}(a-2)^2\pi^{a-\frac{3}{2}}\sin\left(\frac{\pi a}{2}\right)\Gamma(1-a)}{(a-1)}. \tag{A24}$$

Therefore, for $2 < a < 3$ we see that the Fourier transform of the correlation function behaves as

$$C(f) \propto f^{a-3}. \tag{A25}$$

If $a = 3$, then the instances of the Gamma function appearing on the RHS of equation (A22) diverge. Therefore, for $a = 3$ we need to use a different series expansion around $f \to 0_+$. Namely,

$$E_3(2\pi i f) = \frac{1}{2} - 2i\pi f + \pi^2 f^2(2\log(2i\pi f) + 2\gamma - 3) + O\left(f^5\right), \tag{A26}$$

where γ stands for the Euler's constant. The substitution of equation (A26) into equation (A20) leads to

$$\begin{aligned} C(f) = -\mathrm{Re}\Big\{&\left[2\log(\pi f) - 2\gamma + 3 - \log(4)\right]/\left[\sqrt{2\pi}(2i\pi f\log(\pi f)\right. \\ &+ \pi f(2i\gamma + \pi + i(\log(4) - 3)) - 2) \\ &\left.\times(\pi(3i - 2i\gamma + \pi)f - 2i\pi f\log(2\pi f) - 2)\right]\Big\}, \end{aligned} \tag{A27}$$

and thus

$$\begin{aligned} C(f) = \Big(&-8\gamma^3\pi^2 f^2 - 2\pi^2 f(f(-6\log(\pi)(\log\left(16\pi^3\right) \\ &- 2\gamma\log\left(4\pi f^2\right)) + \left(12\gamma^2 + \pi^2\right)\log(\pi f) \\ &+ 9(3 - 4\gamma)\log(2\pi f)) \\ &+ 4f\log^3(f) + 6f(2\gamma - 3 + \log(4) + 2\log(\pi))\log^2(f) \\ &+ 6f\left(\gamma\log(16) + (\log(2\pi) - 3)\log\left(4\pi^2\right)\right)\log(f) \\ &+ 4f\log(2\pi)((\log(2) - 3)\log(2) \end{aligned}$$

$$
\begin{aligned}
&+\log(\pi)\log(4\pi))-4\log(2\pi f))\\
&-4\gamma^2\pi^2 f^2(\log(64)-9)\\
&-2\gamma(\pi^2 f(f\left(\pi^2+27+12\log^2(2)\right)\\
&-4)+4)+\pi^2 f\left(f\left(27-\pi^2(\log(4)-3)\right.\right.\\
&\left.+\log(8)\log(16))-12)-8\log(2\pi f)+12\right)\\
&\Big/\left(\sqrt{2\pi}(4\pi^2 f^2\log(\pi f)(\log(4\pi f)+2\gamma-3)\right.\\
&+\pi^2 f(f(4\gamma^2+\pi^2+(\log(4)-3)^2\\
&\left.+4\gamma(\log(4)-3))-4)+4)^2\right). \qquad \text{(A28)}
\end{aligned}
$$

As $f \to 0$ we have

$$C(f) \sim |\log(f)| \qquad \text{(A29)}$$

Finally, if $a > 3$, then equation (A20) for $f \to 0$ tends to a constant, and thus, Y_t behaves as white noise. Consequently, if the variance of $\tilde{P}(\tau)$ is finite, then Y_t is for large values of t' is uncorrelated.

To summarise, the spectral density function for $f \ll 1$ is,

$$C(f) \overset{f\ll 1}{\propto} \begin{cases} f^{a-3}, & 2<a<3 \\ |\log(f)|, & a=3 \\ \text{const.}, & a>3 \end{cases}. \qquad \text{(A30)}$$

The inversion of the Fourier (cosine) transform in equation (A30) yields

$$C(t') \propto t'^{2-a}/;\ 2<a\leq 3 \wedge t' \gg 1. \qquad \text{(A31)}$$

Appendix 3. Short Selling

In the following, we show that the results presented in section 4.1 remain robust when short-selling is allowed.

To include short selling we allow the demand of the HFs to become negative. However, due to the leverage constraint $\lambda_t^j \leq \lambda_{\max}$ the minimum demand cannot be less than Poledna *et al.* (2014)

$$D_{t,\min} = (1-\lambda_{\max})W_t^j/p_t. \qquad \text{(A32)}$$

Taking the same steps outlined in Appendix B, we solve the optimisation problem

$$\underset{D_t^j\in[D_{t,\min},D_{t,\max}]}{\operatorname{argmax}} \left\{\mathbb{E}\left[U(\gamma W_{t+1}^j)|I^j\right]\right\}, \qquad \text{(A33)}$$

to obtain the following expression for the optimal demand of the jth HF at time t

$$D_t^j = \min\left\{\max\left[1-\lambda_{\max}, \frac{\log(V/p_t)+\frac{1}{2}\mathrm{Var}\left[\log p_{t+1}|I^j\right]}{a\mathrm{Var}\left[\log p_{t+1}|I^j\right]}\right], \lambda_{\max}\right\}$$
$$\frac{W_t^j}{p_t}. \qquad \text{(A34)}$$

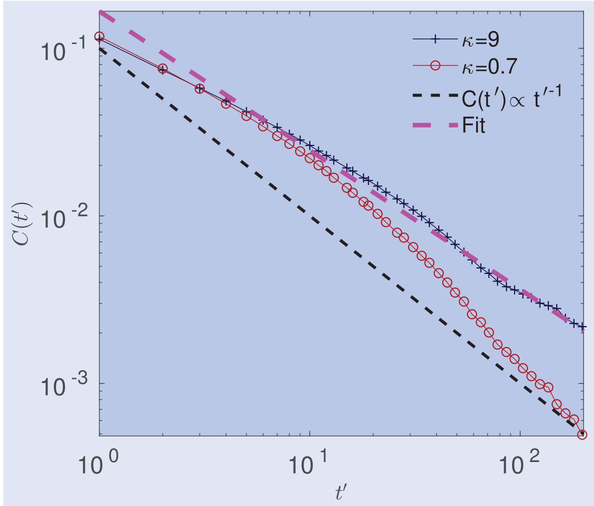

Figure A1. The autocorrelation function of the default symbolic sequence when short-selling is allowed for low ($\kappa = 0.7$) and high ($\kappa = 9$) heterogeneity. The result of fitting the autocorrelation function for $\kappa = 9$ with a power-law model $C(t') \propto t'^{-\eta}$ yields $\eta = 0.88 \pm 0.04$. For the sake of comparison a power-law with exponent -1 (the maximum in absolute terms exponent for defaults to be clustered) is also plotted (black dashed line).

In section 4.1, we showed that for sufficiently high heterogeneity HFs' defaults, as observed on the aggregate level, are clustered. In quantitative terms, this corresponds to the autocorrelation function C of the default sequence being non-summable over the time-lags t', i.e. decaying at most as fast as $C(t') \propto t'^{-1}$.

In figure A1, we present the numerically computed autocorrelation function of the default sequence for two values of $\kappa = 0.7,\ 9$ corresponding to low and high heterogeneity, respectively. The results were obtained by iterating the model described in section 3 [replacing equation (7) with equation (A34) to take into account short selling] for 3×10^7 time steps, and averaging over an ensemble of 200 (random) realisations of the heterogeneity. As can be seen the ACF corresponding to a high degree of heterogeneity decays approximately linearly when plotted on double logarithmic scale. Therefore, the ACF for $\kappa = 9$ decays algebraically, i.e. $C(t') \propto t'^{-\eta}$. Fitting the ACF with a power-law model yields $\eta = 0.88 \pm 0.04$ [$R^2 = 0.9929$], hence the defaults are clustered. In contrast, for lower heterogeneity $\kappa = 0.6$ the decay of the ACF is faster than $C(t') \propto 1/t'$ as can be readily seen by comparing the corresponding ACF (red open circles) with the black dashed line.

Let us now turn our attention to the default statistics. As explained in section 4.2, the clustering of defaults can be linked to an aggregate PDF of waiting times between defaults characterised by infinite variance. In figure A2, we present the corresponding PDFs for $\kappa = 0.7,\ 9$ in double logarithmic scale. As can be seen, for high heterogeneity (blue line), the PDF asymptotically ($\tau \gg 1$) decays as a power law. Fitting the tail of the distribution with a power-law model $\tilde{P}(\tau) \propto \tau^{-\zeta}$ we find $\zeta = 2.88 \pm 0.04$ ($R^2 = 0.9968$) (dashed magenta line). Therefore, the sample variance indeed diverges, i.e. $\zeta < 3$. It is worth noting that the exponents characterising the decay of the ACF and the PDF are in line with the prediction of theorem 1, i.e. $\eta = \zeta - 2$.

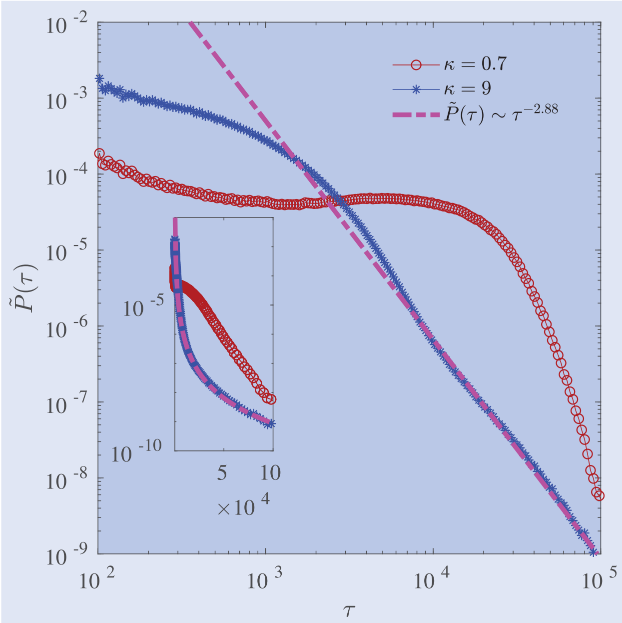

Figure A2. The aggregate PDF of waiting times between defaults for low [$\kappa = 0.7$] and high [$\kappa = 9$] heterogeneity on a double logarithmic scale. Fit with the model $\tilde{P}(\tau) \sim \tau^{-\zeta}$ we obtain $\zeta = 2.88 \pm 0.04$ ($R^2 = 0.9968$). To illustrate the approximate exponential asymptotic decay of the aggregate PDF for $\kappa = 0.7$ we also show the corresponding aggregate densities using a logarithmic scale on the vertical axis (inset).