Uniqueness in Planar Endogenous Business Cycle Theories*

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Abstract

We examine some uniqueness theorems concerning the attractors (limit cycles) of dynamic planar models of non-linear, endogenous theories of business cycle. We confine our attention to the pioneering models of Goodwin, Kaldor, Hicks and their variations. For Goodwin's non-linear multiplier-accelerator model with a single non-linearity, we provide sufficient conditions for establishing uniqueness of the limit cycle based on a theorem by de Figueiredo. We also discuss issues concerning the algorithmic decidability of the number of attractors for these models within the framework of computable analysis.

Keywords: Endogenous business cycles, uniqueness, Goodwin, decidability, one-sided oscillator

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1 Introduction

Economic dynamics has been an important theme in the theoretical pursuits right from the classical economists. Attempts to incorporate dynamic phenomena, in particular, sustained fluctuations into economic theory began getting prominent around the turn of the twentieth century. The corpus of economic theory prevailing then was predominantly static and tied to the notion of equilibrium. These attempts arguably culminated in the birth of modern macroeconomics, in particular business cycle theory, monetary theory, theory of economic policy and growth theory. Developments to this end saw a major surge in theoretical innovations, particularly those in mathematical applications, especially in the 1930s and the immediate decades that followed the second world war. Major contributors include Wicksell, Fisher, Frisch, Kalecki, Tinbergen, Keynes, Hayek, Myrdal, Lindhal, Hawtrey, Aftalion, Schumpeter, Hicks, Harrod, Samuelson, Hansen, Leontief, Solow and Goodwin, among others.

If we focus our attention to business cycle theory, it is perhaps useful to classify different visions to reconcile dynamic phenomena into static equilibrium theory as two types: exogenous and endogenous theories. The exogenous view relied on factors outside the system that disturb an economy which is in equilibrium or steady state as the major driver of fluctuations. In this view, studying business cycles translates to understanding how positive or negative stochastic shocks to a variable, for example technology, translates to output or employment fluctuations. This view has been quite influential in the empirical and policy front till today in orthodox macroeconomic theory in the form of impulse response investigations.

The endogenous tradition in cycle theories, on the other hand, focus on the structure of relationships between different economic variables *within* the capitalistic economic systems. There are many different strands under this broad view. Overall, the nature of relationships between economic variables are such that they make the system prone to sustained fluctuations, even if they are insulated from the disturbances. Exogenous shocks may very well have an impact in the endogenous view, however, they play a subordinate role at best. They are not central to explaining the persistent cyclical tendencies of the economic system. The unifying theme is that the source of these fluctuations are from within the system, which does not have any self-regulating mechanisms to bring itself back to a stable equilibrium or continue to evolve without cycles.

In the development of endogenous cycle theories in the mathematical mode, the presence of non-linear relationships between different variables proved to be a crucial ingredient. Consequently, formalisms from the theory of non-linear oscillations (in the formative years) and non-linear dynamical systems theory (in the later years) were utilized for building mathematical models. They demonstrate different long and short-term properties of the system and their capacity to oscillate by resorting to the application of different existence theorems. In particular, Poincaré–Bendixson theorem and Levinson-Smith theorem were widely used to demonstrate persistent fluctuations in the form of limit cycles.[1] In addition to the existence of limit cycles, questions concerning the number of such limit cycles are also important. If more than one such attractor is present, it is always possible for an economy to head towards one of the undesirable attractors. In such cases, there may a need to steer it away from these basins with the aid of policy. From an algorithmic perspective, whether these mathematical objects (i.e., limit cycles) that are proved to exist and their properties (e.g., their uniqueness), are computable is also a relevant

[1] See Ragupathy and Velupillai (2012).

question.

Stefano Zambelli has been a deep scholar in the endogenous business cycle tradition, especially in studying them via careful numerical approximations and simulations. Together with Vela Velupillai, he has also advanced a computable approach to studying economic dynamics. We both have been fortunate enough to learn these issues from him and later work with him on several aspects within this broad research programme.

In this paper, we examine some of the uniqueness theorems employed in business cycle theories. We confine our attention to the Non-linear, Endogenous theories of Business cycle (NETBC), in particular, to the pioneering models of Goodwin, Kaldor, Hicks and their variations. We focus on the uniqueness proofs concerning the attractors (limit cycles) in these models. Section 2 provides a survey of different uniqueness theorems that were used in Kaldor's trade cycle model. In Section 3 Goodwin's non-linear cycle model is considered and we apply a sufficiency theorem for the non-linear accelerator model, with just one non-linearity. We point out the connection that this theorem has with Goodwin's own contribution. Section 4 addresses the issues concerning the algorithmic decidability of properties of attractors and uniqueness in particular. To this end, we use the framework of computable analysis that provides one way to pose decidability questions for continuous time models.

2 Uniqueness proofs in NETBC

In the planar models of NETBC, the qualitative nature of the attractors that underpin these theories is fairly obvious, viz, limit cycles[2]. Important pioneering models in this tradition are Goodwin (1951), Kaldor (1940), Hicks (1950), Lundberg (1937). These models were broadly in the Keynesian tradition. Their unifying thread was the presence of non-linearities in how different economic variables were related. For example: relationship between income, savings and investment; presence of limits to investment and growth due to natural economic constraints such as full employment. This non-linearity played a crucial role in explaining the observed, sustained fluctuations in aggregate variables such as output and employment. Mathematically, what these economic theories sought to explain (viz., sustained fluctuations of aggregate economic variables over time) were translated to demonstrating the presence of periodic solutions at a local or global level[3]. These theories were formulated in terms of models using differential or difference equations or dynamical systems. The early models that were formulated in continuous time were mostly reduced to one or the other special case of the Liénard equation[4] – van der Pol equation (in the case of Kaldor's model) or the Rayleigh equation Goodwin (1951)), which were known to possess stable periodic solutions. Later models were formulated in terms of dynamical systems and the attention shifted to demonstrating the existence of sustained oscillations by means of existence proofs.Theorems such as Poincaré- Bendixson theorem were used to establish the necessary and sufficient conditions for the presence of limit cycles.

Compared to the use of existence proofs in NETBC, studies providing results concern-

[2]There is also the case of 'centers', which is associated with the growth cycle model of Goodwin (1967)

[3]However, this may not be applicable in the case of Lundberg(1937).

[4]Liénard equation is written as

$$\ddot{x} + f'(x)\dot{x} + g(x) = 0$$

ing the number of possible attractors have been relatively few. The proof of existence and uniqueness was established in some of the early models models by invoking the Levinson-Smith theorem. While Poincaré- Bendixson guarantees the existence of *at least* one limit cycle for planar dynamical systems, the Levinson and Smith theorem establishes the sufficient conditions under which a Liénard equation (a special case of a second-order differential equation) can have a unique isolated periodic solution(limit cycle). There are a couple of observations that may be relevant here. First, uniqueness theorems that are used often provide only sufficient conditions and not the necessary and sufficient conditions for the presence of a unique limit cycle. Presupposing that a cycle already exists for a given system, these theorems provide the conditions for such a cycle to be unique. Therefore, proof of existence is provided first and then these sufficiency conditions are provided for the strip in which the limit cycle exists. Secondly, it is relatively easier to provide sufficient conditions, compared to proving the existence of a limit cycle. The more general mathematical problem concerning the upper bound on the number of limit cycles for a planar polynomial vector field (as a function of the degree of the polynomial) concerns the second part of Hilbert's 16^{th} problem. This problem remains unresolved till today. Consequently, a 'complete' characterization of the nature and number of attractors, even for Liénard equation (which is a special case of the planar polynomial vector fields), is still beyond reach.

2.1 Uniqueness of limit cycle - Kaldor's Model

In the early mathematical models of NETBC, the proof of uniqueness of the limit cycle in the planar models has involved reducing the dynamic model to a generalized Liénard equation and invoking the Levinson-Smith theorem, which provides sufficient conditions for the existence and uniqueness of limit cycles. Depending on the way in which one approximates the economic assumptions into a mathematical model, the number of limit cycles can vary. Therefore, any categorical statement regarding the presence of unique limit cycle must be evaluated in the light of the approximation involved.

We provide a survey of the studies that employ uniqueness theorems in NETBC. We restrict our attention only to analytical proofs for establishing uniqueness and therefore we will not focus on studies which use numerical simulations and other approximate methods. In case of Kaldor's model, the issue of the uniqueness of attractors has been taken up for the different versions of the model by Ichimura (1955), Lorenz (1987) and Galeotti and Gori (1989) . Earliest application of sufficient conditions to guarantee a unique limit cycle was by Yasui (1953) who applied Levinson-Smith theorem to his version of Kaldor's model. In this case, the model was reduced to a van der Pol type equation, which is a special case of the Liénard equation.

Theorem 1 (Levinson Smith Theorem) *[Gandolfo (2005), p.440]*
Consider a two-dimensional differential equation system

$$\dot{x} = y - f(x)$$
$$\dot{y} = -g(x)$$

which is represented as a second-order differential equation,

$$\ddot{x} + f'(x)\dot{x} + g(x) = 0$$

The above equation has a unique periodic solution if the following conditions are satisfied.

1. *$f'(x)$ and $g(x)$ are C^1*

2. *$\exists x_1 > 0$ and $x_2 > 0$ such that for $-x_1 < x < x_2 : f'(x) < 0$ and ≥ 0 otherwise.*

3. *$xg(x) > 0 \; \forall x \neq 0$*

4. *$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} G(x) = \infty$ where $f(x) = \int_0^x f'(s)ds$ and $G(x) = \int_0^x g(s)ds$*

5. *$G(-x_1) = G(x_2)$*

Ichimura (1955) discussed the possibility of applying the above theorem to a particular case of his model, in which he attempted to synthesize the theory of Kaldor, Goodwin and Hicks. However, he perceptively noted that the symmetry condition ($G(-x_1) = G(x_2)$) may not hold for his system and therefore uniqueness of the limit cycle was not certain. Lorenz (1987) later took up the Kaldor model and explicitly addressed the question of uniqueness. He observed that Kaldor's model (as formalised by Chang and Smyth (1971)) does not reduce to a generalized Liénard equation, and therefore Levinson and Smith theorem cannot be applied. In particular, he argued the assumption concerning symmetry of $G(x)$ may be too restrictive from an economic point of view. He contended that it is not possible to apply the above theorem unless one of the following assumptions are made: Investment is assumed to be independent of capital stock or that changes in capital stock is entirely determined by savings, which is independent of the level of capital. This is because in the Kaldor model, as long as one assumes investment function is dependent on both capital stock and income, the resulting second-order differential equation is not a Liénard equation. The main argument was that it was not possible to retain all the assumptions of the original economic model, without making further simplifying assumptions to satisfy mathematical requirements of the theorems invoked.

In reply to this, Galeotti and Gori (1989) contended that it is possible to make use of other uniqueness theorems for Kaldor's trade cycle model and demonstrate the presence of unique limit cycle. By making convenient assumptions and appropriate transformations, they show that Kaldor's system is reducible to a Liénard system.

$$\begin{aligned} \dot{x} &= y - F(x) \\ \dot{y} &= -g(x) \end{aligned} \tag{1}$$

The existence of a limit cycle is proved by using the theorem by Fillipov and they provide the sufficiency conditions for uniqueness of the limit cycle for this model through the following theorem.

Theorem 2 (Zhang Zhi-Fen[5]) *Suppose the system (1) above satisfies the following conditions:*

1. *There exists $a \geq 0$ such that, $F_1(z) \leq 0 \leq F_2(z)$ for $0 \leq z \leq a$, $F_1(z) \not\equiv F_2(z)$ for $0 < z \ll 1$, $F_1(z) > 0$ for $z > a$; $F_2'(z) \leq 0$ for $z \in \{z > 0 | F_2(z) < 0\}$*

2. *$F_1'(z)$ is non-decreasing for $z > a$*

3. *if $F_1(z) = F_2(u)$ with $a < z < u$, then $F_1'(z) \geq F_2'(u)$*

Then the system has at most one limit cycle, which, if exists, must be stable.

The above theorem is utilized by applying the Fillipov transformation[6].

However, in the process of doing so, they overlook the essential message that there is a tendency among economists to force their economic intuitions to the demands of the existence-uniqueness theorems.

Goodwin (1951) did not 'prove' existence or uniqueness of the limit cycle by invoking theorems of these kind, instead he demonstrated the presence of single stable limit cycle (for an S-shaped characteristic) by using geometric methods (Liénard integration). This was later addressed by Sasakura (1996), pp.1771-73:

> In Goodwin's (1951) model there has been something mysterious to business cycle theorists, since, in spite of the simple structure, the question: "Does the model have really a unique stable limit cycle?" could not be solved in general circumstances. In this paper I gave a correct answer to the question: 'Yes, as was expected.' The model has a unique stable limit cycle in an economically meaningful region.

His demonstrated the uniqueness of the limit cycle by using the following theorem, which provides the sufficient conditions.

Theorem 3 *Luo Ding-Jun Uniqueness Theorem*

For the system

$$\dot{x} = y - F(x)$$

$$\dot{y} = -x$$

if $F'(x) - F(x)/x \geq 0$ (or ≤ 0) for all $x \neq 0$, and ***in the strip where the limit cycle exists*** *the left side of the above formula is not identically zero, then the system has at most one limit cycle. (Ye et al. (1986), 139-140)*

The above theorems comprehensively cover almost all the uniqueness results that were employed within this tradition.

3 Uniqueness of Limit cycle and Goodwin's (1951) model

There has been a fair amount of ambiguity about the number of limit cycles that are associated with the Goodwin model. On the one hand, Sasakura points out the 'academic belief' that was held about the presence of a unique limit cycle for this model. Flaschel(2009) also resonates the same belief for his version of Goodwin's model, but at the same time makes a puzzling remark[7] that there could be multiple limit cycles:

[6]Fillipov transformation is done as follows:

$$\begin{array}{ll} z_1 = \int_0^x g(t)dt & \text{if } x \geq 0; F_1(z_1) = F(x) as\ x \geq 0 \\ z_2 = \int_0^x g(t)dt & \text{if } x \leq 0; F_2(z_2) = F(x) as\ x \leq 0 \end{array}$$

z_i is the integral curve of $g(x)$ and therefore instead of the trajectory of (1) in the right and left half-planes, due to the transformation, one deals with the integral curves $\frac{dz}{dy} = F_i(z) - y, i = 1, 2$. Uniqueness in this case is established by way of an comparison argument involving $F_1(z)$ and $F_2(z)$ for $z > 0$.

[7]In chapter 3 footnote 75, Flaschel's remark about uniqueness may be slightly misleading and his reference to Ye et al. (1986) is inaccurate. It is misleading because the uniqueness theorems discussed in Ye et al. (1986) deal with 'sufficiency' conditions and not 'necessary' conditions.

> ... **We state without proof that a [Goodwin] system such as the one we depicted earlier will not only have a closed orbit, but will in fact exhibit just one limit cycle, which will be a globally stable attractor for all other trajectories** of this dynamical system in the above depicted domain.
>
> For economic purposes, **it is, however, not necessary to have a unique and stable limit cycle under all circumstances**. Figure 3.13 shows that all points close to the boundary of the box as well as points close to the stationary state of the dynamics cannot lie on the limit cycle, but will be attracted by one (not necessarily the same) in the interior of the box.
> - Flaschel (2009, p. 97–98, emphasis added).

In this section, let us take a closer look at Goodwin's model in the light of the above discussion on the uniqueness of limit cycle and hope to clarify some of these ambiguities.

3.1 Goodwin's Non-linear Model

Let us briefly sketch the business cycle model developed by Goodwin (1951). In this model, cyclical fluctuations result from the interaction of the dynamic multiplier with a non-linear accelerator. Let $y, \alpha, \dot{k}, \beta, \epsilon$ be the income, marginal propensity to consume, change in capital stock, autonomous consumption and the lag in consumption for the changes in income, respectively. The multiplier relation can be written as,

$$y = \frac{1}{1-\alpha}(\beta + \dot{k} - \epsilon\dot{y})$$

By introducing a lag between the time at which investment outlays are made and their realization (θ - the time to build parameter), we have

$$(1-\alpha)y(t+\theta) + \epsilon\dot{y}(t+\theta) = O_A(t+\theta) + O_I(t+\theta)$$

where,

$$O_A(t+\theta) = \beta(t+\theta) + l(t+\theta)$$

and

$$O_I(t+\theta) \simeq O_D \simeq \psi(\dot{y})$$

Here O_A is the sum of autonomous investment and consumption outlays and O_I is the induced investment $\psi(\dot{y})$ (the nonlinear accelerator or the flexible accelerator). This investment function is assumed to be non-linear.[8] We then have

$$(1-\alpha)y(t+\theta) + \epsilon\dot{y}(t+\theta) = O_A(t+\theta) + \psi(\dot{y}(t)) \tag{2}$$

This we shall call as the *Canonical Goodwin Equation*. From here, based on the kind of approximation we would like, we can have different final equations and consequently, the nature and the number of attractors can vary.

Taking the *Canonical Goodwin Equation* and approximating the equation through Taylor-series expansion for the $(t+\theta)$ terms and retaining only the first two terms of y

[8] Goodwin also introduces a lag in investment and decision outlays - the time lag between decisions on investment and actual investment outlays. However, it is assumed that $O_I(t+\theta) \simeq O_D(t)$

and $\dot{y}$ variable[9], we have:

$$\epsilon\theta\ddot{y} + [\epsilon + (1-\alpha)\theta]\dot{y}] - \varphi(\dot{y}) + (1-\alpha)y = O_A(t+\theta) \quad (3)$$

This is a second-order non-linear difference-differential equation. Goodwin shifts the time co-ordinate of the autonomous injections by θ units and arrives at,

$$\epsilon\theta\ddot{y} + [\epsilon + (1-\alpha)\theta]\dot{y}(t)] - \varphi(\dot{y}) + (1-\alpha)y = O_A(t) \quad (4)$$

The delay term here can be viewed as a periodic forcing to the differential system by the injection of autonomous investment. This equation corresponds to a Rayleigh-type equation with external forcing[10] and to the best of our knowledge, there is no general result characterizing the attractors completely or establishing that this system has a unique limit cycle. As in the case of the forced van der Pol equation, results are known only for some special cases of the forced Rayleigh type equation. However, Goodwin himself went on to further approximate this equation by assuming that O_A to be constant O^* and redefined the variable y in terms of z, representing deviations from the equilibrium value $\frac{O^*}{(1-\alpha)}$. The system is assumed to have a cubic characteristic, i.e, the non-linear accelerator is assumed to be a S-shaped function, with two non-linearities representing the built-in economic constraints.

$$\epsilon\theta\ddot{z} + [\epsilon + (1-\alpha)\theta]\dot{z}(t)] - \varphi(\dot{z}) + (1-\alpha)z = 0 \quad (5)$$

To this, he adds the requirement that the equilibrium is locally unstable and that $d\varphi(0)/d\dot{z} > \epsilon + (1-\alpha)\theta$. By defining the following variables, $x = \sqrt{\frac{1-\alpha}{\epsilon\theta}}z/\dot{z}_0$ and $t_1 = \sqrt{\frac{1-\alpha}{\epsilon\theta}}t$, the above equation can be reduced to a dimension-less form,[11], to the following equation.[12]

$$\ddot{x} + \chi(\dot{x}) + x = 0 \quad (6)$$

$$where, \chi(\dot{x}) = \frac{[\epsilon + (1-\alpha)\theta]\dot{z}(t)] - \varphi(\dot{z})}{\sqrt{(1-\alpha)\epsilon\theta}}$$

Goodwin states that

> Consequently the system oscillates with increasing violence in the central region, but as it expands into the outer regions, it enters more and more into an area of positive damping with a growing tendency to attenuation. It is intuitively clear that it will settle down to such a motion as will just balance the two tendencies, although proof requires the rigorous methods developed by Poincaré.
> ... Perfectly general conditions for the stability of motion are complicated and difficult to formulate, but what we can say is that any curve of the general shape of $X(\dot{x})$ [or $\varphi(\dot{y})$] will give rise to **a single, stable limit cycle**.
> Goodwin (1951, pp.13-14,emphasis added.)

He uses the graphical integration method of Liénard, a geometric method and not a proof of existence and uniqueness, to establish the presence of a limit cycle. However, whether

[9] That is, without taking $O_A(t+\theta)$ term into account.
[10] In this case, constant and periodic.
[11] Refer Goodwin (1951) pp. 12-13.
[12] $\dot{z}_0$ is any unit to measure velocity.

there will be 'single, stable limit cycle' for more complicated functional forms of non-linear accelerator is not addressed.

We can rewrite the above system as

$$\begin{aligned} \dot{x} &= u - \Theta(x) \\ \dot{u} &= -x \end{aligned} \tag{7}$$

where,

$$\Theta(x) = \sigma\tau(x)$$

$$\sigma = \frac{1}{\sqrt{(1-\alpha)\epsilon\theta}}$$

and

$$\tau(x) = [\epsilon + (1-\alpha)\theta]\dot{x}(t)] - \varphi(\dot{x})$$

Does this system have a unique limit cycle? The answer to that question depends on the approximation of the non-linear investment function. For example, Matsumoto (2009) shows that assuming $\varphi(\dot{x}) = vtan^{-1}(x)$(an odd function), the system can have single or multiple limit cycles depending on the values of θ and the local instability condition $\epsilon + (1-\alpha)\theta - v < 0$ on $\Theta(x)$.

Approximating the non-linear accelerator $\varphi(\dot{x}) = vtan^{-1}(x)$ by $x - \frac{1}{3}x^3$ would have different result on the number of limit cycles as opposed to taking the higher order terms, by say the fifth order expression, where $\tau(x) = x - \frac{5}{3}x^3 + \frac{2}{5}x^5$, which has two limit cycles.[13] For the above Liénard form representation of the Goodwin model, the number of limit cycles for the system will depend on the degree of Θ function. Therefore, approximations play a crucial role and it is also necessary to have these approximations correspond to the actual shape of the investment function and compatible with the economic assumptions.[14] Goodwin was aware of this and notes:

> Finally, it should be noted that, while I have assumed a particular shape for $\varphi(\dot{y})$, the power of the Liénard construction is shown by the fact that an equation containing any given curve may be easily integrated. Therefore, whatever sort of investment function is found actually to hold, that type may be completely analyzed in its cyclical functioning. If we look closely into this problem, we find that what is really necessary is to take individual account of many different industries because, while one industry may still have excess capacity, another may be short of fixed capital. Therefore, the combined operation may depend as much on the points at which different industries fire into investment activity as on the actual shape of the X function for each industry or any conceivable aggregation of all of them.
> -(Goodwin, 1951, p. 17)

Instead of approximations of the investment functions, if one resorts to higher order approximations around the time lag parameter θ, it would result in multiple limit cycles as well. Over all, the uniqueness of the limit cycle in Goodwin's model and its independence from initial conditions is not so straightforward.

[13]For $\sigma = 0.08$. Refer Matsumoto (2009) for the details of the approximation.

[14]For example, Matsumoto notes that the fifth order approximation specified above does not have asymptotic bounds that correspond to the ceiling and the floor and the same goes for the values of θ being small or large.

3.2 Sasakura's proof

Sasakura (1996) starts his analysis by taking the following equation mentioned in the last section:

$$\epsilon\theta\ddot{Y} + [\epsilon + (1-\alpha)\theta]\dot{Y}(t)] - \varphi(\dot{Y}) + (1-\alpha)Y = \beta + l \tag{8}$$

He then replaces it with the following, mathematically equivalent system of the previous equation:

$$\begin{aligned} \dot{Y} &= (1/\epsilon)\left(I - (1-\alpha)y + \beta\right) \\ \dot{I} &= (1/\theta)\left(\varphi[\dot{y}(t)] + l - I\right) \end{aligned} \tag{9}$$

Here $I = \dot{K}$ and income is not in terms of deviations from its equilibrium value, but its absolute value. By restricting the domain of income and investment values to an economically meaningful region, he proved the existence of a limit cycle for this region using the Poincaré- Bendixson theorem. He considers the dimensionless form (eq. 6)

$$\ddot{z} + \zeta'(z)\dot{z} + z = 0 \tag{10}$$

Let $u = \dot{z}$ and $v = -z$

$$\dot{u} = v - \zeta(u)$$

$$\dot{v} = -u$$

and uses a theorem that provides sufficient conditions for uniqueness of the limit cycle for this system. Since this theorem does not require any assumptions regarding symmetry, the functional form of the induced investment (or the characteristic) need not be restricted to be symmetric around the origin. From the economic point of view, the meaning of this asymmetry is quite clear.

Theorem 1 (Luo Ding-Jun Theorem) *For the system*

$$\dot{x} = y - F(x)$$

$$\dot{y} = -x$$

if $F'(x) - F(x)/x \geq 0$ (or ≤ 0) for all $x \neq 0$, and ***in the strip where the limit cycle exists*** *the left side of the above formula is not identically zero, then the system has at most one limit cycle. (Ye et al., 1986, p. 139-140)*

The intuitive meaning of the condition is the following: if the marginal propensity to invest due to an increase in income is less than the average propensity in the strip where the limit cycle exists, such a cycle will be unique, within that strip.

3.2.1 One Sided Oscillator

Since the approximations and the assumptions that go into defining the shape of the investment function are crucial in Goodwin's model, we shall examine alternative considerations regarding the shape of non-linear accelerator. The non-linear accelerator in Goodwin (1951) is motivated by the fact that aggregate capital accumulation in an economy cannot go on unhindered forever, since the built-in constraints of the economic system come in to play at some point. In the multiplier-accelerator mode, this meant that the operation of the accelerator mechanism (changes in investment as induced by changes in income) is restricted during the upswing by having reached the desired level of capital (which is a function of income) or by hitting full employment of economic resources. In the downswing, the fact that net investment in fixed capital cannot be lower than the amount required corresponding to rate(s) of depreciation poses a natural lower bound. These constraints have traditionally been dubbed as the ceiling and the floor of the economic system. This means that there are two economically plausible non-linearities, or two plausible bends on the either ends of the accelerator if we wish, in the model of the economy. In Hicks' model, exogenous, autonomous growth factors are superimposed on to a disequilibrium model of fluctuations which incorporates both the ceiling and the floor. Goodwin(1951) model also had two built-in constraints on either side for the accelerator, therefore, utilizing two non-linearities in order to explain sustained oscillations. In terms of the mathematical structure, this means that the dimensionless form (eq. 6) of the reduced master equation has a cubic characteristic.

However, the presence of two bounds is not a necessary condition for showing persistence of cycles in this class of planar models. It was persuasively argued by Goodwin in his review of Hicks' book that economic intuition would suggest that *either* the ceiling (full employment) *or* the floor (lower limit on disinvestment, which is zero) would be sufficient to guarantee the presence of sustained oscillations.[15] In terms of the mathematical structure, this meant that a single non-linearity would be enough to have sustained oscillations, which until then was not thought to be possible. This was shown to be a mathematically plausible by Goodwin himself and thus was born the 'one-sided oscillator'.[16]

Now, if we approximate the investment function with a piecewise linear function, with only one non-linearity - either a ceiling or a floor, it is still possible to show the presence of a limit cycle.[17] Let us consider the case where the accelerator becomes inflexible after having reached a ceiling[18]:

$$\phi(\dot{y}) = \begin{cases} \kappa\dot{y} & \text{if } \kappa\dot{y} \leq \dot{k}^U \\ \dot{k} & \text{if } \dot{y} > \kappa\dot{k}^U \end{cases}$$

Here κ is the accelerator co-efficient and $\dot{k}^U$ is the investment level once the system reaches the ceiling. This can be appropriately modified in case one wants to shift the focus to the floor and on the downswing. Let us measure income in terms of the deviations from its equilibrium value and express

$$\tau(\dot{z}) = \begin{cases} -[\kappa - (\epsilon + (1-\alpha)\theta)]\dot{z} & \text{if } \dot{z} \leq \dot{k}^U/\kappa \\ -[\dot{k}^U - (\epsilon + (1-\alpha)\theta)]\dot{z} & \text{if } \dot{z} > \dot{k}^U/\kappa \end{cases}$$

[15]He notes:

> Either the "ceiling" or the "floor" will suffice to check and hence perpetuate it. Thus the boom may die before hitting full employment,but then it will be checked on the downswing by the limit on disinvestment. Or again it may, indeed it ordinarily does, start up again before eliminating the excess capital as a result of autonomous outlays by business or government. Goodwin (1950, p.319).

[16]A detailed discussion of this discovery can be found in Velupillai (1998).

[17]See: Sordi (2006), which focuses on the simulation aspects of the cycle more than the proof of uniqueness.

[18]Note that this ceiling, given by desired capital level given the income level, need not necessarily coincide with the full employment ceiling. This ceiling can come into effect even before the full employment ceiling is reached.

When we substitute this and following the time translations $x = \sqrt{\frac{1-\alpha}{\epsilon\theta}} z/\dot{z}_0$ and $t_1 = \sqrt{\frac{1-\alpha}{\epsilon\theta}} t$, we arrive at the eq. 5 and reduce it to the dimensionless[19] form, the characteristic of which is as follows:

$$\ddot{x} + \zeta(\dot{x}) + x = 0$$

where,

$$\zeta(\dot{z}) = \frac{\tau(\dot{x}\dot{z}_0)}{\dot{z}_0\sqrt{(1-\alpha)\epsilon\theta}}$$

The above dimensionless equation can be rewritten as:

$$\begin{aligned} \dot{x} &= -u - \zeta(x) \\ \dot{u} &= x \end{aligned} \tag{12}$$

The proof of existence is given by a theorem by de Figueiredo, making use of the Poincaré-Bendixson theorem. (de Figueiredo, 1960, Theorem 1, p. 274).

Theorem 2 ***de Figueiredo's Existence Theorem***
Consider the system (12) above and let

1. $\zeta(0) = 0$
2. $\zeta'(0)$ *exists and is negative and provided there exists a* $y_0 > 0$ *such that*
3. $\zeta(\dot{x}) > 0$, *min*$(\dot{x} \geq y_0)$
4. $2 > -min\zeta'(\dot{x}) < \zeta'(-\dot{x}), (\dot{x} \leq -y_0)$ *except for values of* $\dot{x}$ *at which* $\zeta'(\dot{x})$ *undergoes simple discontinuities.*

Under the above conditions, the system has ***at least*** *one periodic solution.*

Given the above conditions for existence, the sufficient conditions for uniqueness of the periodic orbit are provided by the following theorem:

Theorem 3 *de Figueiredo's Uniqueness Theorem*
Suppose the above system satisfies the above conditions for existence and therefore has a periodic solution. Let there exist a $y_1 > 0$ *such that following conditions hold:*

1. $\zeta(y_1) = \zeta(0) = 0$
2. $\dot{x}\zeta(\dot{x}) < 0, (0 < |\dot{x}| < y_1)$
3. $\zeta(\dot{x}) > 0, (\dot{x} > y_1)$
4. $\zeta'(\dot{x}) \geq \frac{1}{\dot{x}}\,\zeta(\dot{x}), \dot{x} < 0, \dot{x} > y_1$ *except at values of* $\dot{x}$ *where* $\zeta'(\dot{x})$ *undergoes simple discontinuities. Then the system has a unique periodic solution, except for translations in t.*

[19] The piecewise linear characteristic above can be approximated by a function to smoothen the discontinuity (Refer Le Corbeiller (1960)) and we have the following second order differential equation.

$$\ddot{z} - \rho(2 - e^{-\dot{z}})\dot{z} + z = 0 \tag{11}$$

Under the instability condition assumed by Goodwin, i.e, $\kappa > \epsilon + (1-\alpha)$ and for appropriate parameter values, we can establish the presence of sustained oscillations.

A more general theorem for uniqueness is given in de Figueiredo (1970) for a generalized Liénard system.

Theorem 4

$$\dot{x} = y - F(x)$$
$$\dot{y} = -g(x)$$

Suppose the above system has a periodic solution. Let $xg(x) > 0$ *for* $x \neq 0$ *and the following conditions hold for* g, G *and* F*:*

1. f *and* g *are real valued functions which are* C^1 *(Lipschitz condition, which in turn, guarantees the local uniqueness of the solution of the system)*

2. $\lim_{x\to 0}[g(x)/x]$ *exists and is* $\neq 0$

3. $G(x) \to \infty$ *as* $x \to \pm\infty$

4. $2G(x) + y^2 - F(x)y \neq 0 \ \forall (x, y) \neq (0, 0)$

If the inequality $2G(x)f(x) - F(x)g(x) \geq 0$ *holds on the interval* $x < 0, x > x_0$*, where* x_0 *is a positive constant such that* $xF(x) < 0$ *on* $0 < |x| < x_0$*,* $F(x) > 0$ *on* $x > x_0$ *and* $G(x_0) = G(-x_0)$*, then the above system has a unique periodic orbit (except for time translations along* t *axis).*

Remark 5 *For* $g(x) = x$*, the above inequality reduces to* $f(x) - F(x)/x \geq 0$*. Note that this is the same condition that ensures the uniqueness of periodic orbit in Sasakura's use of Luo Ding-Jun's theorem.*[20]

Some remarks here may be pertinent. It is legitimate to wonder whether we are merely providing yet another sufficiency theorem that is applicable to one of the nonlinear models of business cycle, then may be some justification is warranted. This is not meant to be an exercise in showing that a certain sufficiency theorem can be applied, by searching in the compendium of results on sufficiency conditions for unique limit cycles. The motivation is the contrary - to show that economic intuition ought to come first, as in the case of Goodwin's review of Hicks' book. The economic intuition that the accelerator was dead during the downswing motivated to Goodwin to come up with the one-sided oscillator as a mode of clarifying his economic intuition. This was acknowledged by de Figueiredo and Le Corbeiller. de Figueiredo provided sufficient conditions for the unique

[20] On Sasakura's theorem, Sordi (2006) remarks:

> ".. a good starting point is the recent contribution by Sasakura (1996), where the existence of a unique stable limit cycle in Goodwin's model (for the general case of asymmetric nonlinearity of the investment function) is rigorously proved."

Although de Figueiredo's work and Goodwin's role in the discovery of the one-sided oscillator are mentioned, this study overlooks the fact the sufficient conditions in Sasakura's theorem that she discusses are in fact the same conditions (see above) that de Figuerido obtained in his thesis, where the one-sided oscillator played a crucial role. Sasakura also mentions about the one-sided oscillator, but does not discuss similarities of the sufficient conditions. Moreover, in footnote 3, Sasakura (1996) notes that "one of the referees suggested another easier method for proving at least the uniqueness and stability parts as follows: This is to compute the derivative of the Poincaré map and show that (2) is hyperbolic and orbitally asymptotically stable. Then uniqueness drops out easily too.". This is the way de Figuerido proves uniqueness in his 1970 paper.

limit cycles as early as 1958. The economic interpretation of these sufficient conditions are exactly the same as the ones which we seem to have been 'rediscovered' after almost 40 years! In addition, the aim was also show that a more general, economically grounded, yet parsimonious explanation for the persistence of the cycle can be provided within the framework of Goodwin's non-linear cycle model. If one wishes to prove uniqueness and existence and so on, it is possible, but that however was not the concern for Goodwin. In our view, he seemed to be much more interested in unearthing the nature of the cycle.

Proving existence and uniqueness which has preoccupied the theorists of NETBC seems to have limited the potential that non-linear theories of the cycle hold. This meant not having to dig deep to uncover the nature of the cycles, their properties, amplitude, frequency etc., instead merely proving that there are cycles, without providing any explicit method either to find or analyse them. Goodwin's disinterest in proving existence may have been partly because he was concerned with solving or simulating them (geometrically). In contrast, the approximations and simplifications in NETBC often got trapped to the practice of reducing models to equations with known results on uniqueness.

> .. we discuss three different accounts of the original model derived from alternative assumptions...
> In each case the corresponding dynamics is written in Lienard form, **so as to apply a classical result of A. Fillipov** and a more recent theorem of the Chinese mathematician Zhang Zhi Fen.
> ...Indeed in business-cycle Kaldor's systems, which can be driven to Lienard form, several periodic orbits are ruled out: even with an imperfect knowledge of the initial state, the limit cycle to which the economy will eventually tend is univocally determined.
> Galeotti and Gori (1989, pp. 137-38, emphasis added.)

Rather than exploring ways to generalise the models, for example to higher dimensions, removing first approximations and so on, the above quote clearly captures the way in which NETBC modelling activities were directed.

> Lorenz repeatedly underlines the 'ad hoc' character of these further assumptions and their lack of economic meaning ... it is very important, in our opinion, to underline the fact that the empirical relevance of systems presenting a certain number of limit cycles cannot be deduced from the 'realism' of the formal conditions, which are known to guarantee such a dynamical morphology.
> In particular, in the case discussed, if the mathematical hypotheses employed in proving existence and uniqueness of a limit cycle do not appear economically justifiable, this cannot imply that models reducible to Lienard(sic) form, which exhibit a unique periodic orbit, are necessarily 'unrealistic'.
> Galeotti and Gori (1989, p. 137)

Expecting realism of formal conditions is one thing, demanding that the economic assumptions be modified to suit these requirements is quite another. This practise always came at the cost of resorting to ad-hoc assumptions which compromised the rich economic intuitions in these models. Galeotti and Gori assume that either savings or investment is a function of income level alone and independent of the level of capital stock. But this does not escape the criticism that was posed by Lorenz that these assumptions are economically restrictive. These simplifications are done so that the model is reducible to the general Liénard form, so that already available theorems can be readily applied. Even when moving on to higher dimensions and other generalizations, the same temptation to trail behind mathematical results is likely to prevail, for example, in the use of bifurcation theory.

4 Algorithmic decidability and uniqueness

We will now to examine some issues related to computation in the context of dynamical systems and differential equations, which are widely used for modelling economic dynamics. This has implications for computational approaches to studying endogenous economic dynamics as well. We will largely focus on the methodology of computational dynamics in the context of continuous time models or flows.

By computational or algorithmic models, we refer to algorithms in the sense in which they are formally understood in computability theory. This should not be confused with the use of numerical procedures to solve economic models. The focus of the algorithmic approach is therefore on computation, the kind of numbers and processes that are involved in it. If we choose to retain continuous time models and solve them using numerical methods, we need to understand the scope and limitations of this approach. In order to pose questions concerning computation, we first need to provide computational meaning (in the sense in which 'computability' was defined by Alan Turing) to continuous time dynamic models in macroeconomics. By imposing a computability structure, we can ask whether questions about their long-run dynamic properties can be answered algorithmically. Questions concerning the extent of predictability of future events and states of the economy arise naturally in this context. A subset of questions that are interest include the following:

1. Is it possible to characterize the attractors of dynamic economic models algorithmically? Are their domains of attraction computable?

2. Given knowledge of the attractors, can we 'decide' whether and when the economy would reach a given attractor?

3. Can we 'decide' the number of attractors algorithmically?

This brings us to the interface between economic theory, computability theory and dynamical systems. First, almost all the theoretical models in economic dynamics are defined on real number domains and the functions are real valued functions, whereas the mathematics of digital computers deals with numbers and functions that are defined on natural or rational numbers. It is also worth noting that not all real numbers are computable – when we simulate these models on digital computers, we are in fact dealing with functions that are defined over natural, rational or algebraic numbers. Second, the natural data-type of the economic system are themselves rational numbers, at best. Many results that are valid for dynamics defined over real domains do not easily carry over to dynamics defined over rational numbers. The functions over these numbers also need to be equipped with computational content. There are several ways to attempt to bridge the gap between the world of discrete and that of the continuous.

In order to provide a computational structure to dynamical systems, we appeal to definitions and results from *computable analysis* so as to keep the discussion relevant to continuous time models that are widely used in economic dynamics. That means, while talking about computation, we do not impose restrictions such as that the numbers have to be natural or rational (or computable reals) alone, as in classical computability theory. Instead, we resort to computation over real numbers and topological spaces using *type-2* machines. There are several important properties in dynamics (and also other parts of economic theory) that are algorithmically undecidable (see Velupillai (2009)). Since we are interested in endogenous models of economic dynamics and the role of non-linearity, the emphasis will be on non-linear dynamical systems and the algorithmic undecidabilities associated with them. We also discuss the decidable fragments of these models and the assumptions that may be necessary (or sufficient) for them to be decidable. We do not address questions related to computational complexity here. We provide a brief introduction to the notion of computability in continuous time models in the following subsection. We then discuss decidability of attractors, in particular about the decidability of the number of attractors, which is related to uniqueness.

4.1 Continuous-time models and computability

Many models in endogenous business cycle theories are formulated in continuous time and we need to endow them with computational content. Only then can the questions that we raised in the previous section be framed as decision problems concerning economic dynamics. In a decision problem, we are interested in a 'yes' or 'no' decision using an *effective procedure*. Typically, this involves a problem that has many individual sub-problems and one looks for a general method or procedure to answer each of those problems. For example, to decide whether an arbitrary Diophantine equation is solvable is a more general problem with countably infinite sub-problems. Instead of addressing the solvability of each specific problem, one looks at whether there is a general method to decide. If there is no such method available, then each problem or a sub-class of problems might need a specific decision procedure. Some of these sub-problems might not be decidable as well.

Before proceeding, we need to formally define what we mean by a 'procedure', which leads us to the definition of an algorithm, provided by the work of Turing, Church and others. The conventional definition of an algorithm in computability theory is over discrete mathematical objects. This can be viewed as a theory for (discrete) computation on words in some alphabet Σ.[21] According to the Church-Turing thesis, the set of intuitively computable functions are precisely those that are computable via Turing machines. Turing machines themselves are discrete dynamical systems in their own right, and the operation of a Turing machine can be viewed as the evolution of a corresponding discrete dynamical system. Since most models of the economy are formulated in terms of continuous-time dynamical systems that are often defined over (compact) metric spaces, we need a way to tackle this problem for continuous time systems.

For continuous time systems, there is a need for a bridge between the structure on which traditional computability theory is defined ($\mathbb{N}$, which are countable) and continuous systems (defined on sets in $\mathbb{R}^n, n = 1, 2..$ for example, which are uncountable). Notions such as enumerable, recursively enumerable, co-recursively enumerable, semi-decidable, recursive are central to understanding the notion of computability of the functions defined

[21] In a digital computer, one can think of an alphabet as being composed of {0,1}. But it need not necessarily be restricted to binary alphabets and can be generalized to more expressive ones. See Collins (2010).

over sets.

Definition 6 ***Recursively Enumerable Set***
A set $S \subset \mathbb{N}$ is said to be recursively enumerable if and only if it can be accepted by a Turing machine.

Definition 7 ***Recursive Set***
A set $S \subset \mathbb{N}$ is recursive if and only if both the set S and its complement S^c are recursively enumerable.

The idea of enumerability deals with listing the elements of a set, while recursive sets have the property that the problem of membership is solvable for them. There are at least two possible ways to extend the notion of computation to continuous time dynamic models. The first is to extend the theory of computability to continuous-time systems. In classical computability theory, the set $S \subset \mathbb{N}$ and we need to extend it to subsets of $\mathbb{R}$ and functions defined on these sets. The second is to simulate the continuous time model using continuous-time analog machines. The former is the subject of computable analysis or recursive analysis, which extends computability to continuous objects. On the other hand, the analog computation of continuous time models has a fairly long and established history, even within economics. See Velupillai (2011) for a detailed account of this tradition.[22]

In computable analysis,'representations' or naming systems provide a way to link continuous objects (such as real numbers, continuous functions) to other objects that have an explicit computational content or meaning. It is instructive to think of this as a code word that links elements in one domain to those in another. Using these representations, we can induce computability on sets, and the results of the computation can be interpreted in light of these representations. This allows to carry out accurate computations with arbitrary finite precision.

When dealing with real numbers, we require an infinite amount of information to describe an object exactly. We also need to be able to reliably approximate these infinite objects in $\mathbb{R}$ using finite information, and use them for computation or find other ways to overcome this problem. It is possible to extend the traditional notion of computability on words of an alphabet ($\Sigma \to \Sigma$) to sequences of words on an alphabet($\Sigma^\omega \to \Sigma^\omega$). Since real numbers can be represented via infinite sequences, computability can now be defined for mappings between infinite sequences. This is referred to as Type-2 computability or Type-2 effectivity (see Weihrauch (2000), Ch:2 for more details), where infinite sequences act as representations for a real number. Note that Type-2 computability is still explicitly based on Turing computability and it is only as powerful. The infinite amount of computation associated while dealing with infinite inputs and outputs (sequence of decimals for example) can be finitely approximated to any desired level of precision in this framework and can be simulated using digital computers.

[22] One of the well known theoretical models of a universal continuous-time analogue machine is perhaps Claude Shannon's General Purpose Analog Computer (GPAC). The computational power of computable analysis and GPAC is shown to be equivalent, at least in the case of real computable functions over compact domains (Bournez et al. (2007)). However, there seems to be no explicit agreement on the class of computable functions via different models of analog machines, as opposed to the case of digital computation, where we have the Church-Turing thesis.

These representations can be extended to topological spaces as well (Weihrauch (2000), Brattka and Weihrauch (1999)), and concepts such as effective or computable topological spaces, admissible representations and names, computability over real numbers, computability over closed, open and compact sets, can be appropriately defined (Chapters 2-5, Weihrauch (2000)). This approach relies heavily on the nexus between continuity and computability of functions. Please refer to the appendix for some basic definitions concerning computable analysis that are utilised in this paper.

Definition 8 *1. A sequence $\{r_n\}$ of rational numbers is called a ρ-name of a real number x if there are three functions a, b and c from $\mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$, $r_n = (1)^{a(n)} \frac{b(n)}{c(n)+1}$ and $|r_n - x| \frac{1}{2^n}$*

2. A double sequence $\{r_{n,k}\}_{n,k\in\mathbb{N}}$ of rational numbers is called a ρ-name for a sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers if there are three computable functions a, b, c from $\mathbb{N}^2 \to \mathbb{N}$ such that, for all $k, n \in N$, $r_{n,k} = (1)^{a(k,n)} \frac{b(k,n)}{c(k,n)+1}$ and $|r_{n,k} - x_n| \frac{1}{2^k}$

3. A real number x (a sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers) is called computable if it has a computable ρ-name, i.e. there is a Type-2 machine that computes the ρ-name without any input.

Definition 9 ***Computable Functions***
Let A, B be sets, where ρ-names can be defined for elements of A and B. A function $f : A \to B$ is computable if there is a Type-2 machine such that on any ρ-name of $x \in A$, the machine computes as output a ρ-name of $f(x) \in B$.

Definition 10 *1. An open set $E \subseteq \mathbb{R}^m$ is called **recursively enumerable** (r.e. for short) open if there are computable sequences $\{a_n\}$ and $\{r_n\}, a_n \in \mathbb{Q}^m$ and $r_n \in \mathbb{Q}$, such that $E = \bigcup_{n=0}^{\infty} B(a_n, r_n)$.*

*2. A closed subset $K \subseteq \mathbb{R}^m$ is called **r.e. closed** if there exist computable sequences $\{b_n\}$ and $\{s_n\}, b_n \in \mathbb{Q}^m$ and $s_n \in \mathbb{Q}$, such that $\{B(b_n, s_n)\}_{n\in\mathbb{N}}$ lists all rational open balls intersecting K.*

*3. An open set $E \subseteq \mathbb{R}$ is called **computable** (or recursive) if E is r.e. open and its complement E^c is r.e. closed. Similarly, a closed set $K \subseteq \mathbb{R}^m$is called computable (or recursive) if K is r.e. closed and its complement K^c is r.e. open.*

*4. A compact set $M \subseteq \mathbb{R}^m$is called **computable** if it is computable as a closed set and, in addition, there is a rational number b such that $\| x \| \leq b \forall x \in M$.*

4.2 Attractors of dynamic economic models and computability

Having extended the notion of computability to real numbers and topological spaces via representations, we can now turn to dynamic economic models formulated in these spaces. It should be remembered that these extensions do not enable us to compute more than what is computable by a Turing machine. This should not be interpreted as advocating or dismissing the use of continuous time economic models. Rather, we are interested in exploring what properties can be declared computable in this class of economic models. In macroeconomic dynamics, given certain assumptions regarding the relationships between different economic variables, we are interested in exploring and characterizing certain long term properties of an economic system such as steady state paths, equilibrium points, limit cycles, periodic or chaotic attractors and their stability properties. If more than one attractor is possible, then we need understand the conditions under which the system tends to one or the other, i.e, their respective domains of attraction. This can be viewed as the characterization of the ω limit set associated with a given dynamic economic model. In the case of endogenous economic dynamics, the focus will be on the algorithmic characterization of equilibrium points and periodic attractors, as has been the major focus of this tradition.

In the context of aggregate economic dynamics, we may ask: given a representation of an economy as a (non-linear) dynamical system, can we algorithmically characterize features concerning the periodic attractors. It turns out that many long-run properties associated with these models are, in general, undecidable. Therefore, exhaustive algorithmic classification of attractors is generally not possible. Since this is not a big surprise for the case of continuous time models, we focus on a class for which these properties are known to be decidable. Fortunately, the major class of economic models of endogenous dynamics – which happen to be planar systems – do seem to have important properties that are algorithmically decidable under certain stability conditions. In non-linear models, Poincaré-Bendixson theorem helps to classify the attractors on the plane. In higher dimensions, the attractors can be highly complicated and we do not have general results for classifying them as we do on the plane. Some formal definitions concerning dynamical systems and related notions are provided in the appendix.

Definition 11 ***Dynamical System***
A dynamical system describes the evolution of points on a state space over time. Let X is an open subset of $\mathbb{R}^n$. A dynamical system on X is a C^1 function

$$\phi : \mathbb{R} \times X \to X$$

, where $\phi_t(x) = \phi(t, x)$ and ϕ_t satisfies the following conditions:

1. *$\phi_0(x) = x$ for all $x \in X$*
2. *$\phi_t \circ \phi_s(x) = \phi_{t+s}(x)$ for all $s, t \in \mathbb{R}$ (in case of discrete time systems $s, t \in \mathbb{N}$) and $x \in X$.*

The evolution rule ϕ is a map (for discrete-time system) or can be written as a differential equation (for a continuous-time, C^1 system).

Definition 12 ω ***limit set***
Let $\phi(t, x)$ be the flow of the above dynamical system and z be a point on this trajectory. z is called a ω limit point of the trajectory of the dynamical system if there exists a sequence $t_n \to \infty$ such that $lim_{n\to\infty}\phi(t_n, x) = z$. The ω limit set of x ,$\omega(x)$, is the set of all ω-limit points $z \in X$

Definition 13 ***Invariant Set***
A set $L \subset \mathbb{R}^n$ is called an invariant set if $\phi(t, x) \in L$, for all $x \in L$ and $t \to \infty$.

Definition 14 ***Attracting Set***
The closed invariant set L is called an ***attracting set*** *of the flow $\phi(t,x)$ if $\exists$ some neighbourhood V of L, such that, $\forall x \in V$ and $\forall t \geq 0, \phi(t,x) \in V$ and $\phi(t,x) \to L$ as $t \to \infty$*

Definition 15 ***Domain of Attraction*** *Domain of attraction of the attracting set L of $\phi(t,x)$ is defined as,*

$$\Theta_L = \bigcup_{t \leq 0} \phi_t(V)$$

the union of all neighbourhoods V of the attracting set, for which $\forall x \in V$ and $\forall t \geq 0, \phi(t,x) \in V$ and $\phi(t,x) \to L$ as $t \to \infty$

For now, let us assume that we know the qualitative properties of the possible attractors for the economic model. This is what is usually done when one invokes the Poincaré-Bendixson theorem to prove the existence of limit cycles. Computability and decidability questions about the ω-limit set can be analysed in different ways. One is to compute the attracting set - such as fixed points and limit cycles, explicitly. The second is to algorithmically decide whether a trajectory belongs to the domain of attraction of a given attractor. There is also the issue of explicitly computing the domain(s) of attraction for members of the ω-limit set.

When we ask such questions about attractors, we are essentially asking whether the set is recursive. By providing a computable structure, we endow the set with the property of recursive enumerability. That is, there is a rule or an algorithm or a partial recursive function to list the successive members of this set. In order for this set to be decidable, we require an algorithm that will decide whether a given element belongs to the set or not in finite time. It is intuitive that all recursive sets are recursively enumerable, but the converse is not true. By establishing a correspondence between the economy formulated as a dynamical system and a Turing machine, we can study the dynamic trajectories of an economy via the evolution of a Turing machine.

4.3 Decidability of attractors in planar models

Planar, non-linear models form an important class of models in the tradition of endogenous business cycle theory. What is the status of these models when it comes for algorithmic decidability and computability of attractors? Although several of their properties are undecidable, there are some decidable fragments. Graça and Zhong (2011) conclude that attractors and their basins of attractions are *semi-computable* if we assume that the system is stable. They work within the framework of computable analysis and type-2 machines.

Definition 16 ***Semi-Computable Functions***
A function $\psi : A \to O(\mathbb{R}^m)$, (where $O(\mathbb{R}^m) = \{O | O \subseteq \mathbb{R}^m$ is open in the standard topology$\}$ is called ***semi-computable*** *if there is a Type-2 machine such that on any ρ-name of $x \in A$, the machine computes as output two sequences $\{a_n\}$ and $\{r_n\}$, $a_n \in \mathbb{Q}^m$ and $r_n \in \mathbb{Q}$, such that $\psi(x) = \bigcup_{n=0}^{\infty} B(a_n, r_n)$.*

Definition 17 ***Type-2 Machine*** *[Weihrauch(2000), p.15]*
Let Σ^* be the set of finite words over some arbitrary finite alphabet Σ. Similarly, let Σ^ω be the set of infinite sequence of symbols from some arbitrary finite alphabet Σ, which has at least two elements. A Type-2 Machine M is a Turing machine with k input tapes together

with a type specification $(Y_1, Y_2....Y_k, Y_0)$ *with* $Y_i \in (\Sigma^*, \Sigma^\omega)$*, giving the type for each input tape and the output tape.*

Since the attractors cannot be computed in general, we need to explore the conditions under which they become computable. Stability becomes a crucial condition for ensuring the computability of attractors. They employ the notion of computability on closed, open and compact sets as outlined earlier, following the work of Brattka and Weihrauch (1999) and Weihrauch (2000).

Theorem 18 *Let* $x' = f(x)$ *be a planar dynamical system. Assume that* $f \in C^1(\mathbb{R}^2)$ *and that the system is structurally stable. Let* $K \subseteq \mathbb{R}^2$ *be a computable compact set and let* K_{cycles} *be the union of all hyperbolic periodic orbits of the system, which is contained in* K*. Then, given as input* ρ*-names of* f *and* K*, one can compute a sequence of closed sets* $\{K^n_{cycles}\}_{n\in\mathbb{N}}$ *with the following properties:*

1. $K^n_{cycles} \subseteq K$ *for every* $n \in \mathbb{N}$
2. $K^{n+1}_{cycles} \subseteq K^n_{cycles}$ *for every* $n \in \mathbb{N}$
3. $\lim_{n\to\infty} K^n_{cycles} = K_{cycles}$

This means that, under the assumption of structural stability (together with the Lipschitz property), if one can supply the ρ names of f and the compact set K as input, there is an algorithm which can tell, in finite time, whether f has a periodic orbit of the above dynamical system in the compact set K. Since the periodic orbits are only *semi-decidable* in this case, one may need an infinite amount of time, countably calibrated, to conclude that K does not contain a periodic orbit. The same is true for the equilibrium points of the above dynamical system. This is a necessary consequence of dealing with a recursively enumerable, but not recursive set.

4.4 Decidability of the number of attractors

We are now ready to address the issue of algorithmic decidability of the number of attractors associated with a given planar dynamical system. Deciding whether the attractor (limit cycle in our case) is unique is a special case of this problem. Since the models invoking these theorems often assume compactness, we grant these assumptions and ask whether the number of attractors for a given economy (formulated as a planar dynamical system) is algorithmically decidable. We can consider a situation that is general enough by allowing the function to be analytic.

Proposition 19 *Consider a non-linear model of an economy, formulated as a planar dynamical system* $\dot{x} = \psi(x)$ *on a compact subset* $K \subseteq \mathbb{R}^2$*. Consider the case where* ψ *is described by an analytic function and assume that the system is structurally stable.*

1. *The possible limit sets (hyperbolic equilibria and hyperbolic limit cycles) are necessarily finite.*
2. *Given the* ρ *names of the compact set* $K \subseteq \mathbb{R}^2$ *and the analytic function* ψ*, the problem of deciding the number of equilibria and limit cycles is in general undecidable.*

Proof 20 *The first part follows from Dulac's theorem (Perko (2001), pg. 205-6). The second part follows from Graça and Zhong (2011), theorems 20 & 21.*

Thus we note that within the framework of computable analysis, it not possible to algorithmic decide the number of attractors, in general. The precise number and position of limit cycles of planar polynomial vector fields is the subject of Hilbert's 16^{th} problem, which is yet to be resolved.

5 Conclusion

We examined various uniqueness theorems utilised in the developments that followed some of the pioneering contributions in the non-linear endogenous business cycle models. We have argued that there has been a tendency to straightjacket economic models to fit the mathematical requirements in several cases. We also highlighted some issues concerning the algorithmic decidability of properties of attractors, such as their uniqueness. This brings us to the question of whether a methodology strongly reliant on existence-uniqueness-stability aspects which may have their legitimate place in dynamical systems theory are relevant at all for endogenous economic theories. We believe that an excessive reliance on existence-uniqueness-stability mode of theorising stunted the possibilities for exploring the rich avenues that these endogenous theories possessed. Stefano Zambelli's work on carefully studying the dynamic possibilities of economies within the framework of coupled oscillators through numerical simulations shows us a way to overcome some of these limitations (Zambelli, 2015). It may be entirely possible to devise methods that are suited for the specificities, nuances of the economic system and their evolution as a continued transformation of its own structure. Dynamical systems of the sort analysed earlier have a *given* structure and one searches for associated attractors and their properties. The key aspect of economic dynamics is the continual change in structure, composition/dimension with birth of new products, sectors, technologies and evolving demand patterns.

Zambelli's painstaking development of a theory of production, extending the research of Piero Sraffa in innovative ways, where distribution issues are at the core, has already broken grounds (Zambelli, 2018a,b). His sustained advocacy of computable economic dynamics (together with Velupillai), which pays close attention to the actual nature of the domain of economic quantities, i.e., rational numbers, and algorithms provides promising avenues for further research. In this Zambelli seems to be in elite, enviable company with Alan Turing for his philosophy towards studying economic problems. Turing noted (in the context of morphogenesis) that rigid (classical) mathematical requirements that one adopts at the initial stages of developing a theory ought to give way to more powerful computational methods, even if it is at the cost of an all embracing theory:

> Most of an organism, most of the time, is developing from one pattern into another, rather than from homogeneity into a pattern. One would like to be able to follow this more general process mathematically also. **The difficulties are, however, such that one cannot hope to have any very embracing theory of such processes, beyond the statement of the equations. It might be possible, however, to treat a few particular cases in detail with the aid of a digital computer.** This method has the advantage that it is not so necessary to make simplifying assumptions as it is which doing a

more theoretical type of analysis. ... The essential disadvantage of the method is that one only gets results for particular cases. **But this disadvantage is probably of comparatively little importance**. - Turing (1952, pp. 71-72, emphasis added.)

References

Bournez, O., M. L. Campagnolo, D. S. Graça, and E. Hainry (2007). Polynomial differential equations compute all real computable functions on computable compact intervals. *Journal of Complexity 23*(3), 317 – 335.

Brattka, V. and K. Weihrauch (1999). Computability on subsets of Euclidean Space I: Closed and Compact subsets. *Theoretical Computer Science 219*(1-2), 65–93.

Chang, W. W. and D. Smyth (1971). The Existence and Persistence of Cycles in a Non-linear Model: Kaldor's 1940 Model Re-examined. *The Review of Economic Studies, 38*(1), 37–44.

Collins, P. (2010). Computable Analysis with Applications to Dynamic Systems. *Centrum Wiskunde & Informatica, Working Paper (MAC-1002).*

de Figueiredo, R. (1960). Existence and uniqueness of the periodic solution of an equation for autonomous oscillations. In L. L. Cesari and S. Lefschetz (Eds.), *13, Contributions to the Theory of Nonlinear Oscillations*, Volume 5, (Annals of Mathematical Studies, no. 45), pp. 269–284. Princeton, NJ: Princeton University Press.

de Figueiredo, R. (1970). Existence and uniqueness results for liénard's equation. *IEEE Transactions on Circuit Theory CT-17,*(3), 313–321.

Flaschel, P. (2009). *The Macrodynamics of Capitalism* (second edition ed.). Berlin: Springer.

Galeotti, M. and F. Gori (1989). Uniqueness of periodic orbits in lienard-type business-cycle models. *Metroeconomica 40*(2), 135–146.

Gandolfo, G. (2005). *Economic Dynamics.* Heidelberg: Springer.

Goodwin, R. (1950). A Non-Linear Theory of the Cycle. *The Review of Economics and Statistics 32*(4), 316–320.

Goodwin, R. (1951). The Nonlinear Accelerator and the Persistence of Business Cycles. *Econometrica, 19*(1), 1–17.

Goodwin, R. (1967). A Growth Cycle. In C. Feinstein (Ed.), *Socialism, Capitalism and Economic Growth: Essays Presented to Maurice Dobb*, pp. 54–58. Cambridge: Cambridge University Press.

Graça, D. and N. Zhong (2011). Computability in planar dynamical systems. *Natural Computing 10*(4), 1295–1312.

Hicks, J. (1950). *A Contribution to the Theory of the Trade Cycle.* Oxford: Clarendon Press.

Ichimura, S. (1955). Toward a General Nonlinear Macrodynamic Theory of Economic Fluctuations. In *Post-Keynesian Economics by K. K. Kurihara,*, pp. 92 – 226. London: George Allen & Unwin.

Kaldor, N. (1940). A Model of the Trade Cycle. *The Economic Journal 50*(197), 78–92.

Le Corbeiller, P. (1960). Two-stroke oscillators. *IRE Transactions on Circuit Theory CT-7*(4), 387–398.

Lorenz, H.-W. (1987). Strange attractors in a multisector business cycle model. *Journal of Economic Behavior & Organization 8*(3), 397 – 411.

Lundberg, E. (1937). *Studies in the Theory of Economic Expansion.* P.S.King & Son, London.

Matsumoto, A. (2009). Note on Goodwin's 1951 Nonlinear Accelator Model with an Investment Delay. *Journal of Economic Dynamics and Control 33*(4), 832–842.

Perko, L. (2001). *Differential Equations and Dynamical Systems.* New York: Springer.

Ragupathy, V. and K. Velupillai (2012). Existence Proofs in Nonlinear Endogenous Theories of the Business Cycle on the Plane – The Origins. *ASSRU Discussion Paper DP-10-2012/1.*

Sasakura, K. (1996). The business cycle model with a unique stable limit cycle. *Journal of Economic Dynamics and Control, Vol.20*(9-10), 1763–I 773.

Sordi, S. (2006). 'Floors' and/or 'Ceilings' and the Persistence of Business Cycles. In T. Puu and I. Sushko (Eds.), *Business Cycle Dynamics- Models and Tools.* Berlin: Springer–Verlag.

Turing, M. A. (1952). The chemical basis of morphogenesis. *Philosophical Transactions of the Royal Society of London. Series B, Biological Sciences 237*(641), 37–72.

Velupillai, K. (1998). The vintage economist. *Journal of Economic Behavior and Organization 37*(1), 1–31.

Velupillai, K. (2011). The Phillips Machine and the Epistemology of Analogue Computation. *Economia Politica [Journal of Analytical and Institutional Economics] XXVII*(Special issue), 39–62.

Velupillai, K. V. (2009). Uncomputability and Undecidability in Economic Theory. *Applied Mathematics and Computation 215*(4), 1404–1416.

Venkatachalam, R. (2013). *Studies in Endogenous Macroeconomic Dynamics.* Ph. D. thesis, University of Trento, Italy.

Weihrauch, K. (2000). *Computable Analysis - An Introduction.* Berlin: Springer-Verlag.

Xianwu, Z., Z. Zhi-Fen, and G. Suzhi (1994). On the Uniqueness of the Limit Cycle of the Generalized Liénard Equation. *Bulletin of the London Mathematical Society 26*(3), 213–247.

Yasui, T. (1953). Self-Excited Oscillations and the Business Cycle. *Cowles Commission Discussion Paper No. 2065*.

Ye, Y.-Q., C. Sui-lin, C. Lan-sun, H. Ke-cheng, L. Ding-jun, M. Zhi-en, W. Er-nian, W. Ming-shu, and Y. Xin-an (1986). *Theory of Limit Cycles*, Volume Vol. 66,. Providence, RI: Translations of Mathematical Monographs, American Mathematical Society.

Zambelli, S. (2015). Dynamical coupling, nonlinear accelerator and the persistence of business cycles. *Cambridge Journal of Economics 39*(6), 1607–1628.

Zambelli, S. (2018a). The aggregate production function is NOT neoclassical. *Cambridge Journal of Economics 42*(2), 383–426.

Zambelli, S. (2018b). Production of Commodities by Means of Commodities and non-uniform rates of profits. *Metroeconomica 69*(4), 791–818.

Zhi-Fen, Z. (1986a). On the Number of Limit Cycles of Lienard Equation. *Quaderni dell'Istituto Mathematico U. Dini, Università di Firenze, Firenze A.A:*, 85–86.

Zhi-Fen, Z. (1986b). Proof of the uniqueness theorem of limit cycles of generalized Lienard equations. *Applicable Analysis 23*(1-2), 63–76.