Bipartite post-quantum steering in generalised scenarios

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The study of stronger-than-quantum effects is a fruitful line of research that provides valuable insight into quantum theory. Unfortunately, traditional bipartite steering scenarios can always be explained by quantum theory. Here we show that, by relaxing this traditional setup, bipartite steering incompatible with quantum theory is possible. The two scenarios we describe, which still feature Alice remotely steering Bob's system, are: (i) one where Bob also has an input and operates on his subsystem, and (ii) the 'instrumental steering' scenario. We show that such bipartite post-quantum steering is a genuinely new type of post-quantum nonlocality, which does not follow from post-quantum Bell nonlocality.

Introduction.– Einstein-Podolsky-Rosen steering is a striking nonlocal feature of quantum theory [1, 2]. First discussed by Schrödinger [1], it refers to the phenomenon where Alice, by performing measurements on half of a shared system, remotely 'steers' the state of a distant Bob, in a way which has no classical explanation. From a modern quantum information perspective [2] steering certifies entanglement in situations where Alice's devices are uncharacterised or untrusted, allowing for "one-sided device independent" implementations of information-theoretic tasks, such as quantum key distribution [3], randomness certification [4, 5], measurement incompatibility certification [6–8], and self-testing [9, 10].

Given the usefulness of quantum steering as a resource for information processing, a comprehensive understanding of this non-classical phenomenon as a resource is highly desirable. A fruitful way to approach this, pursued in the study of other non-classical phenomena, e.g. Bell nonlocality [11] and contextuality [12], is to investigate it 'from the outside': namely, to study it operationally from the perspective of a more general theory – which may supersede quantum theory – and then understand which aspects are purely quantum. Studying phenomena beyond what quantum theory predicts is relevant not only from the hypothetical perspective of a post-quantum theory, but also – and above all – because it allows for a deeper understanding of the foundations of quantum theory and the limitations it has for information processing [13]. The main question studied here is how to properly understand steering from this more general perspective, on which we report substantial progress in this Letter.

Abstractly, we may view the steering scenario as one where Alice has a device that accepts a classical input, x, usually thought of as labelling the choice of measurement, and produces a classical outcome, a, usually thought of as the measurement result, while Bob has a device without an input, that produces a quantum system, which is correlated with the input and outcome of Alice, and usually thought of as the steered system. Here we are interested in the possibility that the local structure of quantum theory is maintained, while considering more general global structure – for instance more general types of correlations or global dynamics.

In this setting, we would like to re-examine the phenomenon of steering. A natural question that arises is whether a more general theory may allow for steering beyond what quantum theory predicts. That is, could it be possible to find a pair of devices for Alice and Bob which could not be produced within quantum theory, by Alice and Bob sharing a quantum state, upon which Alice performs measurements labelled by x and with outcomes a? The only requirement that we maintain in this generalised setting is that of relativistic causality: Alice should not be able to use steering to signal to Bob, i.e., to send information to him instantaneously.

A celebrated theorem by Gisin [14] and Hughston, Josza and Wootters [15] (GHJW) shows that post-quantum steering cannot occur in the traditional setting. Namely, any pair of devices that do not allow signalling from Alice to Bob can always be realised by some carefully chosen set of measurements and quantum state. The traditional setting is however not the only interesting scenario where one can see the steering phenomena. In Ref. [16], post-quantum *multi-partite* steering was discovered: in a tri-partite scenario, Alice and Bob are able to jointly steer the state of a third party, Charlie in a way which cannot arise from measurements an any quantum state. Subsequently, unified frameworks for studying quantum and post-quantum steering in the multipartite setting have been developed, providing a playground for exploring this fascinating effect [18, 19].

A key question that nevertheless remained unanswered is whether it is possible to have post-quantum steering in a suitable generalised bipartite scenario, or whether post-quantum steering is a purely multipartite phenomenon. In this Letter we answer this question in the positive. We discover two natural bipartite generalisations of steering that allow for post-quantum effects (see Fig. 1 cases (c) and (d) respectively): one where Bob also has an input that allows him to additionally influence his quantum state, and another where this additional influence is instead conditioned on Alice's outcome. This second generalisation corresponds to a specific type of setup known as the 'instrumental causal network', that is ubiqui-

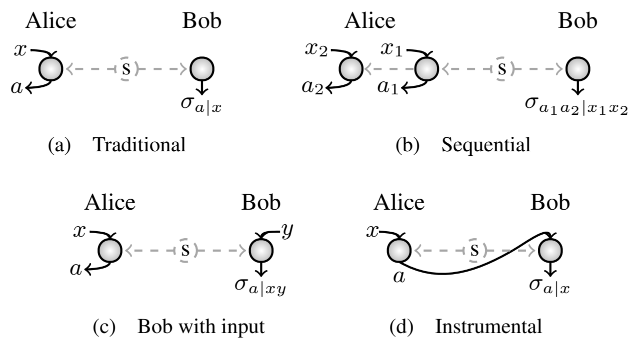

FIG. 1. Different generalised bipartite steering setups (a) The traditional scenario: Alice makes a measurement, steering the state of Bob. (b) The sequential-measurement scenario: Alice now performs a sequence of measurements, steering the state of Bob multiple times (c) The Bob-with-input (BWI) scenario: Bob now also has an input, allowing him to also influence his state, but performing some operation on it (d) The instrumental steering scenario: is similar to BWI except that Bob's input now depends on Alice's outcome. The top two scenarios ((a) and (b)) do not admit post-quantum steering. We show here that the bottom two scenarios ((c) and (d)) have post-quantum steering.

tous in causal inference [20, 21]. Furthermore, we show, crucially, that in both cases the post-quantum steering uncovered genuinely constitute new effects, that are distinct from post-quantum nonlocality in the associated generalised setups. We do this by finding explicit examples of post-quantum steering where if Bob performs measurements on his quantum system, then the resulting outcome statistics are never post-quantum nonlocal.

Preliminaries.— In the traditional bipartite quantum steering scenario (see Fig. 1(a)) Alice and Bob share a system in a possibly entangled quantum state ρ. Alice is allowed to perform generalised measurements on her share of the system, which correspond to positive-operator valued measures (POVM). Alice chooses one such measurement $\{M_{a|x}\}_a$, labelled by x, from a set of measurements, and obtains an outcome a with probability $p(a|x) = \mathrm{tr}\left\{(M_{a|x} \otimes \mathbb{I}_B)\rho\right\}$. After the measurement, Bob's steered state is $\rho_{a|x} = \mathrm{tr}_A\left\{(M_{a|x} \otimes \mathbb{I}_B)\rho\right\}/p(a|x)$. It is convenient to work with the unnormalised steered states $\sigma_{a|x} = p(a|x)\,\rho_{a|x} = \mathrm{tr}_A\left\{(M_{a|x} \otimes \mathbb{I}_B)\rho\right\}$, which contain both the information about both Alice's conditional probabilities $p(a|x) = \mathrm{tr}\left\{\sigma_{a|x}\right\}$, and Bob's conditional states $\rho_{a|x}$. The collection $\{\sigma_{a|x}\}_{a,x}$ of unnormalised states Bob is steered into is called an *assemblage*. Due to the completeness relation for Alice's measurements, $\sum_a M_{a|x} = \mathbb{1}$ for all x, it follows that $\sum_a \sigma_{a|x} = \mathrm{tr}_A\{\rho\} = \rho_B$, independent of x. This can be seen as a no-signalling condition from Alice to Bob, since Bob, without knowledge of the outcome of Alice, has no information about the choice of measurement she made.

One natural generalisation of the traditional steering scenario is to allow Alice to make a sequence of measurements on her share of the system, such that each measurement has the potential to steer the state of Bob. This situation is depicted in Fig. 1 (b). In the supplemental material [22], we show that this generalisation in fact does not feature post-quantum steering either; this can be seen as an extension of the GHJW theorem [14, 15] to the sequential scenario.

Bipartite steering when Bob has an input (Fig. 1c).— We consider now the generalisation where Bob's device also accepts an input before producing a quantum state. Intuitively, we can think that this input may determine the preparation of some quantum system, which could come about from a transformation on a quantum system inside Bob's device. This situation is depicted in Fig. 1 (c), where y denotes the input. In this generalised scenario, the members of the assemblage will be $\{\sigma_{a|xy}\}_{a,x,y}$. Note that when the variable y takes only one possible value, this scenario reduces to the traditional bipartite steering setup of Fig. 1(a).

In the context of quantum theory, we assume that Alice and Bob share a quantum state ρ and that Alice performs measurements labelled by x, as in the standard scenario. Given that Bob now has an input, the most general operation that he could apply is a Completely-Positive and Trace-Preserving (CPTP) channel onto his part of the quantum system. Thus, the quantum assemblages that can be generated are:

Definition 1. ***Quantum Bob-with-input assemblages.*** *An assemblage $\{\sigma_{a|xy}\}_{a,x,y}$ has a has a quantum realisation in the Bob-with-input steering scenario iff there exists a Hilbert space $\mathcal{H}_A$ and POVMs $\{M_{a|x}\}_{a,x}$ for Alice, a state ρ in $\mathcal{H}_A \otimes \mathcal{H}_B$, and a collection of CPTP maps $\{\mathcal{E}_y\}_y$ in $\mathcal{H}_B$ for Bob, such that*

$$\sigma_{a|xy} = \mathcal{E}_y\left[\mathrm{tr}_A\left\{(M_{a|x} \otimes \mathbb{I})\rho\right\}\right]. \tag{1}$$

We denote this set of assemblages as $\mathcal{Q}_{BWI}$.

Note that this definition of Quantum Bob-with-input (BWI) assemblages does not require that the operations take place in a particular order. That is, the same assemblage can be obtained if the map $\mathcal{E}_y$ is applied to Bob's subsystem before Alice measures hers.

To go beyond quantum theory, we have to identify the most general constraints that apply here. Not only must we now ensure no-signalling from Alice to Bob, but since Bob has an input, we must also ensure no-signalling from Bob to Alice. These constraints are captured by the following definition:

Definition 2. ***Non-signalling Bob-with-input assemblages.*** *An assemblage $\{\sigma_{a|xy}\}_{a,x,y}$ is non-signalling in the Bob-with-input steering scenario iff $\sigma_{a|xy} \geq 0$ for all a, x, y, and*

$$\sum_a \sigma_{a|xy} = \sum_a \sigma_{a|x'y} \quad \forall\, x, x', y\,, \tag{2}$$

$$\mathrm{tr}\left\{\sigma_{a|xy}\right\} = p(a|x) \quad \forall\, a, x, y\,, \tag{3}$$

$$\mathrm{tr}\sum_a \sigma_{a|xy} = 1 \quad \forall\, x, y\,, \tag{4}$$

where $p(a|x)$ is the probability that Alice obtains outcome a when performing measurement x on her share of the system. We denote the set of such assemblages as $\mathcal{G}_{BWI}$.

We can now return to our central question of whether there can exist post-quantum steering in this scenario. Here we find that this is indeed the case:

Theorem 1. *The set of all non-signalling Bob-with-input assemblages is strictly larger than the set of quantum Bob-with-input assemblages, $\mathcal{Q}_{BWI} \not\equiv \mathcal{G}_{BWI}$. Hence, there is post-quantum steering in the Bob-with-input steering scenario.*

Proof. We construct an explicit example of a assemblage in $\mathcal{G}_{BWI}$ which cannot be realised in quantum theory.

Consider the specific scenario where Alice has binary inputs and outcomes, $x \in \{0,1\}$ and $a \in \{0,1\}$, Bob has a binary input $y \in \{0,1\}$, and the dimension of Bob's Hilbert space is 2. Consider the following assemblage:

$$\sigma^*_{a|xy} := \frac{1}{2}\left(|a\rangle\langle a|\,\delta_{xy=0} + |a\oplus 1\rangle\langle a\oplus 1|\,\delta_{xy=1}\right) \quad (5)$$

Note that (i) $\sigma^*_{a|xy} \geq 0$ for all a,x,y; (ii) $\sum_a \sigma^*_{a|x,y} = \frac{1}{2}(\delta_{xy=0}+\delta_{xy=1})\mathbb{I} = \frac{1}{2}\mathbb{I}$, which is independent of x and y; (iii) $\mathrm{tr}\left\{\sigma^*_{a|xy}\right\} = \frac{1}{2}(\delta_{xy=0}+\delta_{xy=1}) = \frac{1}{2}$ is independent of y and (iv) $\mathrm{tr}\sum_a \sigma^*_{a|xy} = 1$. This shows that $\{\sigma^*_{a|xy}\}$ is a valid no-signalling assemblage, i.e., $\{\sigma^*_{a|xy}\} \in \mathcal{G}_{BWI}$.

Now we show that this assemblage cannot arise in quantum theory, i.e., $\{\sigma^*_{a|xy}\} \notin \mathcal{Q}_{BWI}$. We do so by first noting that for a quantum-realisable assemblage, since $(\mathbb{I}_A \otimes \mathcal{E}^y)[\rho]$ is a bipartite quantum state when $\mathcal{E}^y$ is a CPTP channel, Alice and Bob can only produce quantum Bell correlations, should Bob choose to measure his system. Namely, let Bob make an arbitrary measurement $\{N_b\}_b$, on his state in a quantum assemblage $\{\sigma_{a|xy}\}_{a,x,y}$. Then, the correlations obtained are

$$\begin{aligned} p(a,b|x,y) &= \mathrm{tr}\{N_b\sigma_{a|xy}\}, \\ &= \mathrm{tr}_B\{N_b\mathcal{E}_y\left[\mathrm{tr}_A\left\{(M_{a|x}\otimes\mathbb{I})\rho\right\}\right]\}, \\ &= \mathrm{tr}\left\{(M_{a|x}\otimes\mathcal{E}_y^\dagger(N_b))\rho\right\}. \end{aligned}$$

Since $\mathcal{E}_y(\cdot)$ is a CPTP channel , the dual map $\mathcal{E}_y^\dagger(\cdot)$ is unital, and hence $\mathcal{E}_y^\dagger(N_b)$ is always valid POVM. This provides an explicit quantum realisation of the correlations $p(a,b|x,y)$. We will thus prove that (5) is not quantum-realisable by demonstrating that it can generate correlations $p(a,b|x,y)$ which are known to be impossible within quantum theory.

Let $N_b = |b\rangle\langle b|$ be the computational basis measurement. The correlations that Alice and Bob obtain are

$$p(a,b|x,y) = \langle b|\,\sigma^*_{a|xy}\,|b\rangle = \begin{cases} \frac{1}{2} & \text{if} \quad a\oplus b = xy \\ 0 & \text{otherwise} \end{cases} .$$

These are the correlations of the "Popescu-Rohrlich" box [23], which are not achievable within quantum theory. Hence, $\sigma^*_{a|xy} \notin \mathcal{Q}_{BWI}$ and so $\mathcal{Q}_{BWI} \not\equiv \mathcal{G}_{BWI}$. □

We see then that post-quantum steering can arise in a generalised bipartite steering scenario. This example given however relies on post-quantum nonlocality and hence the post-quantum steering found may be argued to be just another guise of the former effect. In the following theorem, we prove that the two phenomena are genuinely different:

Theorem 2. *Post-quantum steering in the Bob-with-input steering scenario is independent of post-quantum nonlocality. Namely, there exist non-signalling assemblages $\{\sigma_{a|xy}\}$ that are not quantum realisable, but which can only lead to quantum correlations $p(a,b|x,y)$ in the Bell scenario.*

The proof of this theorem is given in the Supplemental Material [22]. The main idea is to show that the following assemblage has the desired properties, i.e., that it is post-quantum and that whenever Bob performs a measurement $\{N_b\}$ on it, the observed outcome statistics $p(ab|xy) = \mathrm{tr}\left\{N_b\,\sigma_{a|xy}\right\}$ have always a quantum realisation:

$$\sigma_{a|xy} = \frac{1}{4}(\mathbb{1} + (-1)^{a+\delta_{x,2}\delta_{y,1}}\sigma_x) \quad (6)$$

where $x \in \{1,2,3\}$ and $(\sigma_1,\sigma_2,\sigma_3) = (X,Y,Z)$ are the Pauli operators.

The method to show that this assemblage can only yield quantum correlations is to notice that one may mathematically represent this assemblage as Alice performing Pauli measurements on the maximally entangled state, and Bob applying either the identity or transpose map (which crucially is positive but not completely positive) depending on y [17]. Then, following Ref. [18], the assemblage Eq. (6) can only yield quantum correlations. In the Supplemental Material we further provide two alternative proofs that the assemblage is postquantum [22]: one using an argument based around self-testing, and the second by constructing an explicit steering inequality that is (robustly) violated beyond the quantum bound by the assemblage (6). While doing this, in the Supplemental Material, we present a method to bound the quantum bound of a steering inequality in this scenario.

Hence, just as for multipartite post-quantum steering [16], the effect here is independent of the existence of post-quantum Bell nonlocality.

Instrumental steering (Fig. 1d).— We now consider the instrumental steering scenario [24]. In this case, Bob still has an input that can inform the preparation of a quantum system, however now this input can depend on Alice's measurement outcome (see Fig. 1 (d)). For example, Bob's input could just decide a transformation upon a quantum system. To recover the traditional steering scenario, we again enforce the constraint that Bob only has one input, and thus we trivially have no dependence on Alice's output. This scenario is closely related to the so-called 'instrumental setup' [20, 21], only now one of the variables has become a quantum system. Indeed, this close relation between instrumental steering and the instrumental setup, will enable us to identify a connection between the instrumental steering scenario and the Bob-with-input scenario further below.

In the instrumental steering scenario, an assemblage is given by the collection of subnormalised states $\{\sigma_{a|x}\}$, where x denotes the choice of measurement by Alice, and a denotes both Alice's outcome and Bob's input. Within quantum theory, the assemblages they can generate are the following:

Definition 3. ***Quantum Instrumental assemblages.*** *An assemblage $\{\sigma_{a|x}\}_{a,x}$ has a has a quantum realisation in the instrumental steering scenario iff there exists a Hilbert space $\mathcal{H}_A$ and POVMs $\{M_{a|x}\}_{a,x}$ for Alice, a state ρ in $\mathcal{H}_A \otimes \mathcal{H}_B$, and a collection of CPTP maps $\{\mathcal{E}_a\}_a$ in $\mathcal{H}_B$ for Bob, such that*

$$\sigma_{a|x} = \mathcal{E}_a \left[\mathrm{tr}_A \left\{ (M_{a|x} \otimes \mathbb{I})\rho \right\} \right] . \quad (7)$$

We denote this set of assemblages by $\mathcal{Q}_I$.

The instrumental steering scenario has no straightforward non-signalling constraints. Hence, in order to define general assemblages here, we adopt the relation between non-signalling Bell correlations and generic instrumental correlations in the black-box scenario found in Ref. [25] (see also Supplementary Material of Ref. [26]). In the instrumental setup, where the so-called device-independent instrumental correlations are studied, it was recently found that these correlations are indeed a post-selection of the correlations found in a Bell scenario: the post-selection procedure consists on keeping the events where $y = a$ [25]. This inspires the following definition:

Definition 4. ***General instrumental assemblages.*** *An assemblage $\{\sigma_{a|x}\}_{a,x}$ is a general instrumental assemblage iff there exists a non-signalling Bob-with-input assemblage $\{\omega_{a|xy}\} \in \mathcal{G}_{BWI}$ such that $\sigma_{a|x} = \omega_{a|x,y=a}$ for all a and x. We denote the set of such general assemblages by $\mathcal{G}_I$.*

This definition allows us to adopt the viewpoint of Refs. [25, 26], and hence understand the assemblages in the instrumental steering scenario as being a post-selection of those in a Bob-with-input scenario.

Note that this connection between the Bob-with-input scenario and the instrumental steering scenario allows us to interpret the latter beyond the traditional way that the instrumental setup is presented. Usually, the instrumental setup is such that there is signalling from Alice to Bob, since he needs to learn her outcome in order to implement the operation on his system. However, the particular perspective brought in by Ref. [25], and which we adopt here, highlights that, ultimately, this communication plays no distinct role in how resourceful the assemblages are, since the Bob-with-Input scenario does not allow for signalling and can simulate them.

Returning to our central question, we now show that there is post-quantum steering in the instrumental steering scenario. Moreover, we show that this does not follow from post-quantum instrumental black-box correlations, and it is hence another independent form of post-quantumness.

Theorem 3. *The set of general instrumental assemblages strictly contains the set of quantum instrumental assemblages, $\mathcal{Q}_I \not\equiv \mathcal{G}_I$. Hence, post-quantum instrumental steering exists.*

Theorem 4. *Post-quantum steering in the instrumental steering scenario is independent of post-quantum instrumental correlations. Namely, there exist general assemblages $\{\sigma_{a|x}\}$ that are not quantum realisable, but which can only lead to quantum correlations $p(a, b|x)$ in the instrumental scenario.*

These two theorems are proven together in the Supplementary Material, but their proof is very similar to that of Theorem 2. The general assemblage that is used here as an example is that which derives from (6) by setting $y = a$, which is both provably post-quantum in the instrumental scenario, and can only lead to quantum instrumental black-box correlations.

Thus, post-quantum steering is also possible within the instrumental scenario, and this is independent of the existence of correlations with no quantum explanation in the fully device-independent instrumental scenario. Hence, post-quantum instrumental steering is another genuinely new effect. Finally, in terms of number of variables (inputs and outputs), the instrumental scenario is the simplest one where post-quantum steering can exist.

Discussion.— Exploring plausible effects beyond quantum theory that are nevertheless consistent with relativistic causality [23], is important from various perspectives: on the one hand, it allows possible extensions of quantum theory to be explored, in the light of quantum gravity [27]. On the other, it allows us to develop a deeper understanding of quantum theory itself, by identifying those properties of it that are truly quantum [28–31]. Here, we have shown that for the important form of nonlocality known as steering, it is possible in principle to go beyond what quantum theory allows even when considering only two parties, if suitable generalisations of the traditional scenario are considered. Crucially, we showed that our examples of post-quantum steering are genuinely new, and are not related to other post-quantum nonlocal effects.

In addition, on the way, we also showed that post-quantum steering is impossible in the sequential-measurement generalisation of steering, schematically depicted in Fig. 1 (b). As such, we have extended the GHJW no-go theorem [14, 15] to this setting, with detailed provided in the Supplemental Material [22].

The 'instrumental setup' [20, 21] is known to be the one with the fewest number of variables able to admit a classical-quantum gap [32]. This is closely related to the setup of Fig. 1(d), except that Bob's system is a classical variable. Previously, classical-quantum gaps had been found in Bell-type [25, 26, 33] and steering [24] scenarios. Furthermore, quantum-post-quantum gaps have also been found in Bell-type scenarios [25, 26]; but the existence of post-quantum instrumental steering remained an open question. The discovery of the latter here thus also resolves this open question.

Going forward, the most interesting question now is to understand the power of post-quantum steering. For instance, are there information-theoretic or physical principles that are violated by the newly-discovered forms of post-quantum steering found here? In addition, it would be interesting to explore information processing tasks exploiting post-quantum steering as a resource. For example, one task where traditional steering is a resource is subchannel discrimination [34]. It would be interesting to study whether post-quantum steering gives an advantage in tasks related to this. More generally, now that we have uncovered post-quantum steering in a bipartite setting, it paves the way for analysing a broad range of bipartite tasks from this new direction. Indeed, we note that our newly introduced Bob-with-input steering scenario has already been investigated within the context of resource theories [35].

Our overarching hope is that studying quantum theory 'from the outside', whether from the perspective of steering, or other nonlocal and nonclassical effects, will lead to novel insights into the very structure of quantum theory and the possibilities and limitations of quantum theory for information processing. We expect our results and new insights to contribute to this rapidly developing and exciting field.

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[17] Note that the fact that the assemblage can be represented as arising through the application of a postitive but not completely positive map is a useful mathematical fact, which simultaneously provides a route to show that it is post-quantum (since it involves a map which is not allowed in quantum theory), and that it cannot lead to post-quantum nonlocality, due to Ref. [18]. This representation does not dictate that the assemblage has to arise in a such a way; how it would be physically prepared will depend upon details of the post-quantum theory. Our particular way to express the assemblage here is merely one mathematical representation that is useful for our purposes.
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