SIAM J. FINANCIAL MATH.
Vol. 11, No. 1, pp. 148–168

Theory of Cryptocurrency Interest Rates*

Dorje Brody†, Lane Hughston‡, and Bernhard Meister§

Abstract. A term structure model in which the short rate is zero is developed as a candidate for a theory of cryptocurrency interest rates. The price processes of crypto discount bonds are worked out, along with expressions for the instantaneous forward rates and the prices of interest-rate derivatives. The model admits functional degrees of freedom that can be calibrated to the initial yield curve and other market data. Our analysis suggests that strict local martingales can be used for modeling the pricing kernels associated with virtual currencies based on distributed ledger technologies.

Key words. cryptocurrencies, distributed ledger technologies, blockchains, interest rate models, pricing kernels, foreign exchange, derivatives

AMS subject classifications. 91G30, 91G20, 60G99

DOI. 10.1137/19M1263042

1. Introduction.

Over a decade has passed since the release of the white paper establishing bitcoin—the progenitor of the burgeoning open-source cryptocurrency movement [1]. In the intervening years the ability of digital currencies to flourish with market capitalizations in the billions of dollars without the backing of sovereign states has been widely publicized. One of the key factors that has to this date hindered cryptocurrencies from reaching the mainstream and expanding beyond fringe applications to link with the real economy in significant ways, despite numerous preliminary attempts, has been the inability of regulated trading consortiums to develop broad platforms for cryptocurrency derivatives and structured products, and more generally to put in place the robust infrastructure needed to support financial markets analogous to those we take for granted in connection with sovereign currencies. In economies based on well-established sovereign currencies, interest rate derivatives are central to the functioning of financial markets. A statistical bulletin published in December 2018 by the Basel-based Bank of International Settlement illustrates how the notional amount for fixed-income derivatives trumps every other category by a multiple of five or more [2]. For an interest-rate derivatives market to function, products need to be priced, payoffs need to be replicated, and positions need to be hedged—and for pricing, replicating, and hedging, it is essential that financial institutions, regulators, and other market participants should have at their disposal a diverse assortment of serviceable interest-rate models that are well adapted to characteristics of the currencies in which the instruments being traded are based. When it comes to cryptocurrencies, this requirement immediately poses a challenge, since cryptocur-

*Received by the editors May 20, 2019; accepted for publication (in revised form) December 17, 2019; published electronically March 3, 2020.
https://doi.org/10.1137/19M1263042
†Department of Mathematics, University of Surrey, Guildford GU2 7XH, UK (d.brody@surrey.ac.uk).
‡Department of Computing, Goldsmiths College, London SE14 6NW, UK (l.hughston@gold.ac.uk).
§School of Management, Swansea University, Swansea SA1 8EN, UK (bernhard.k.meister@gmail.com).

rencies by their nature offer no short-term interest, and it seems impossible prima facie that one should be able to build an interest rate model for which the short rate is identically zero. In the case of bitcoin, the regularly updated blockchain represents the distributed ledger and therefore holdings in the cryptocurrency. The technical details of the updating process are highly involved, but do not impinge upon the general statement that holdings of bitcoin as recorded in the ledger do not earn interest. No central bank exists. New coins are not issued to existing holders of the coins, but instead are awarded to "miners" successfully solving numerical puzzles necessary to update the blockchain. Other prominent cryptocurrencies provide variations on this general theme of not paying by accommodating additional functionality. For example, ethereum, the second largest cryptocurrency, incorporates distributed computing.

What are the implications for interest rate modeling? To begin, it may be helpful if we recall what is generally meant by a "conventional" interest rate model. This will allow us to identify some of the differences between the conventional theory and the interest rate theory required for cryptocurrencies. Such conventional models exist in abundance and include, for instance, most of the examples mentioned in [3, 4]. One typically assumes the existence of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with an associated filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, upon which an adapted short-rate process $\{r_t\}_{t\geq 0}$ is defined. Here, time 0 denotes the present. By a unit discount bond we mean a financial instrument that delivers a single cash flow of one unit of currency at time T and derives its value entirely from that cash flow. The price P_{tT} of a T-maturity unit discount bond at time $t < T$ is then given by a conditional expectation of the form

$$P_{tT} = \mathbb{E}^{\mathbb{Q}}\left[\, \exp\left(-\int_t^T r_s \,\mathrm{d}s\right) \middle| \mathcal{F}_t\right], \tag{1}$$

taken under a suitably specified risk-neutral measure $\mathbb{Q}$ that is equivalent (in the sense of agreeing on null sets) to the physical measure $\mathbb{P}$. Glancing at (1) one might conclude that it is not possible to have a nontrivial discount-bond system if $r_t = 0$ for all $t \geq 0$, and indeed this is true in "conventional" models. For example, in the HJM model [5], the instantaneous forward rates, which determine the discount bond prices via the relation

$$P_{tT} = \exp\left(-\int_t^T f_{tu}\,\mathrm{d}u\right), \tag{2}$$

for $t < T$, are given by

$$f_{tT} = \frac{1}{P_{tT}}\,\mathbb{E}^{\mathbb{Q}}\left[r_T \exp\left(-\int_t^T r_s\,\mathrm{d}s\right) \middle| \mathcal{F}_t\right]. \tag{3}$$

Thus f_{tT} is the forward price per unit notional, made at time t, for purchase of the rights to a cash flow of the amount r_T per unit notional at time T. It follows that if the short rate vanishes, then so do the instantaneous forward rates, in which case the discount bond system trivializes and takes the form $P_{tT} = 1$ for all $t < T$. The same conclusion follows directly from (1). Thus if the short rate vanishes in a conventional interest rate model it follows immediately that so do all the term rates.

Nevertheless, just because the short rate vanishes we are not necessarily forced to conclude that the discount bond system is trivial. Beginning with the work of Constantinides [6], finance theorists have learned to think about interest-rate modeling in a more general way, in terms of so-called pricing kernels. The pricing kernel method avoids some of the technical issues that arise with the introduction of the risk neutral measure and the selection of a preferred numeraire asset in the form of a money market account, and at the same time it leads to interesting new classes of interest rate models (see, e.g., [7, 8, 9, 10, 11] and references cited therein). By a pricing kernel, we mean an $\{\mathcal{F}_t\}$-adapted càdlàg semimartingale $\{\pi_t\}_{t\geq 0}$ satisfying (a) $\pi_t > 0$ for $t \geq 0$, (b) $\mathbb{E}[\pi_t] < \infty$ for $t \geq 0$, and (c) $\liminf_{t\to\infty} \mathbb{E}[\pi_t] = 0$, with the property that if an asset with value process $\{S_t\}_{t\geq 0}$ delivers a bounded $\mathcal{F}_T$-measurable cash flow H_T at time T and derives its value entirely from that cash flow, then

$$S_t = \mathbb{1}_{\{t<T\}} \frac{1}{\pi_t} \mathbb{E}\left[\pi_T H_T \middle| \mathcal{F}_t\right]. \tag{4}$$

Here we write $\mathbb{E}$ for expectation under $\mathbb{P}$. We adopt the convention that the value of such an asset drops to zero at the instant the cash flow occurs. Thus $\lim_{t\to T} S_t = H_T$, whereas $S_t = 0$ for $t \geq T$. This is in keeping with the usual analysis of stock prices when a stock goes ex-dividend, and respects the requirement that the price process should be càdlàg. Our assumptions imply that $\pi_T H_T$ is integrable and that $\{\pi_t S_t + \mathbb{1}_{\{t\geq T\}} \pi_T H_T\}_{t\geq 0}$ is a uniformly integrable martingale. The existence of an established pricing kernel is equivalent, in a broad sense, to what we mean by market equilibrium and the absence of arbitrage. In fact, it can be shown under rather general conditions [12] that with a few reasonable assumptions any pricing formula for contingent claims takes the form (4). Then if $H_T = 1$, we obtain an expression for the price at time t of a bond that pays one unit of currency at T, given by

$$P_{tT} = \mathbb{1}_{\{t<T\}} \frac{1}{\pi_t} \mathbb{E}\left[\pi_T \middle| \mathcal{F}_t\right]. \tag{5}$$

To be sure, models of the type represented by formula (1) can be obtained as instances of models of the type represented by formula (5), but it is not the case that all interest rate models are of type (1). As we shall demonstrate, models of type (5) can be constructed for which the unit of conventional currency is replaced by a unit of cryptocurrency, and in such a way that we arrive at a nontrivial interest rate model for which the cryptocurrency condition $r_t = 0$ is satisfied for all $t \geq 0$ and yet for which term rates are nonvanishing.

The present paper explores a class of such models achieved by allowing the crypto pricing kernel to be a strict local martingale. The reasoning behind this proposal is as follows. The pricing kernel methodology requires that the price $\{S_t\}$ of a non-dividend-paying asset should have the property that the product $\{\pi_t S_t\}$ should be a martingale. Now, suppose the market admits a unit-initialized absolutely continuous money market account with value process $\{B_t\}$ of the form

$$B_t = \exp\left(\int_0^t r_s \, \mathrm{d}s\right). \tag{6}$$

Then the process $\{\Lambda_t\}$ defined by $\Lambda_t = \pi_t B_t$ is a martingale, and the pricing kernel as a consequence is given by

$$\pi_t = \exp\left(-\int_0^t r_s\,\mathrm{d}s\right)\Lambda_t\,. \tag{7}$$

In that case, if we introduce the Radon–Nikodym derivative defined by

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \Lambda_t\,, \tag{8}$$

we can make a change of measure in (4), and we are led to the well-known risk-neutral valuation formula

$$S_t = \mathbb{1}_{\{t<T\}}\,\mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s\,\mathrm{d}s\right)H_T\middle|\,\mathcal{F}_t\right]. \tag{9}$$

Then if we set $H_T = 1$, we recover the class of "conventional" interest rate models defined by a bond price of the form (1) along with the money market account (6).

In the class of models we consider for crypto interest rates, which will be introduced in section 2, we exclude the existence of an instantaneous money-market asset from the model altogether. This can be achieved by choosing the pricing kernel to be a strict local martingale. This implies that the short rate vanishes and hence that the process $\{B_t\}$ defined by (6) is constant. But if $\{B_t\}$ is constant, then $\{\pi_t B_t\}$ is not a martingale, and thus there is no money-market asset. We consider a model in which the pricing kernel is given by the reciprocal of a Bessel process of order three. This reciprocal process, introduced in [13], is a well-known example of a strict local martingale [14, 15, 16, 17] and has the advantage of being highly tractable. The idea that this process can be used as a pricing kernel appears in [10], where it is recognized that the resulting interest rate model does not admit a representation for the bond price in the form (1). Here we develop a model of this type in detail in the context of cryptocurrency bonds. In section 3 we derive explicit expressions for the discount bond system and the various associated rates. The results are applied in section 4 to obtain pricing formulae for digital options on discount bonds and caplets on simple crypto rates. In section 5, we introduce a class of related models based on Bessel(n) processes with $n \geq 4$, and the fourth order model is worked out in detail. In section 6, we introduce a class of models based on a complexification of the Bessel(3) process. We conclude in section 7 with some remarks about options on crypto exchange rates.

2. A model of no interest.

The pricing kernel formalism allows us to identify where the theory of cryptocurrencies deviates from the conventional one: namely, there is no money market account. But how is it possible to construct an interest rate model without a money market account? This apparent impossibility in the context of a conventional interest rate theory is nonetheless possible in a pricing kernel framework. The argument is as follows. First, we observe, by virtue of (5), that a necessary and sufficient condition for the bond price to be a decreasing function of T for any fixed t such that $t < T$ is that $\{\pi_t\}$ should be a supermartingale. This implies that the interest rate system associated with $\{\pi_t\}$ is nonnegative. Now, cryptocurrencies are by their nature storable assets, with negligible storage costs, and hence by a standard arbitrage argument cannot be borrowed at a negative rate of interest. Thus it is reasonable to assume that the crypto pricing kernel is a supermartingale. In any case, in what follows we shall make that assumption. We recall that a semimartingale is a local martingale if for any increasing sequence of stopping times $\{\tau_n\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty}\tau_n=\infty$, the stopped process is a martingale for each value of n. A strict local martingale then is a local martingale that is not a true martingale. Now, it is well known that a positive local martingale is necessarily a supermartingale. Thus, if we allow for the possibility that the pricing kernel is a strict local martingale, then the positivity of the pricing kernel implies that it is a supermartingale with vanishing drift. This suggests that the cryptocurrency interest-rate term structure can be modeled by letting the pricing kernel be a strict local martingale.

As an illustration, we examine an interest rate model based on a pricing kernel given by the reciprocal of the Bessel process of order three. Specifically, the pricing kernel is constructed as follows. Let $\{W_t^{(1)},W_t^{(2)},W_t^{(3)}\}_{t\geq 0}$ be three independent standard Brownian motions on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and define

$$X_t=\int_0^t\sigma_s\mathrm{d}W_s^{(1)},\quad Y_t=\int_0^t\sigma_s\mathrm{d}W_s^{(2)},\quad Z_t=\int_0^t\sigma_s\mathrm{d}W_s^{(3)},\tag{10}$$

where $\{\sigma_t\}_{t\geq 0}$ is a deterministic function which we take to be bounded and strictly positive. We then define a model for the pricing kernel by setting

$$\pi_t=\frac{1}{\sqrt{(X_t-a)^2+(Y_t-b)^2+(Z_t-c)^2}}\,,\tag{11}$$

where a,b,c are constants, not all equal to zero. The initial condition $\pi_0=1$ requires that $a^2+b^2+c^2=1$, and rotational symmetry implies that the vector (a,b,c) can lie on any point on the unit sphere in $\mathbb{R}^3$. We note that the function $u:\mathbb{R}^3\to\mathbb{R}^+\cup\infty$ defined by

$$u(x,y,z)=\frac{1}{\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}}\tag{12}$$

is a solution of Laplace's equation

$$\frac{\partial^2}{\partial x^2}u(x,y,z)+\frac{\partial^2}{\partial y^2}u(x,y,z)+\frac{\partial^2}{\partial z^2}u(x,y,z)=0\tag{13}$$

on $\mathbb{R}^3\backslash(a,b,c)$. One can think of $\{u(x,y,z)\}$ as the Coulomb potential generated by a unit charge situated at the point (a,b,c). Then we have $\pi_t=u(X_t,Y_t,Z_t)$, and using the fact that the three Brownian motions are independent, we find that the drift of $\{\pi_t\}$ vanishes and that $\{\pi_t\}$ satisfies the dynamical equation

$$\mathrm{d}\pi_t=-\sigma_t\,\pi_t^2\,\mathrm{d}W_t\,,\tag{14}$$

where the process $\{W_t\}_{t\geq 0}$ defined by the relation

$$\mathrm{d}W_t=\pi_t\left[(X_t-a)\mathrm{d}W_t^{(1)}+(Y_t-b)\mathrm{d}W_t^{(2)}+(Z_t-c)\mathrm{d}W_t^{(3)}\right]\tag{15}$$

is a standard Brownian motion. The interpretation of (14) is as follows. Suppose we consider a generic market model in which the pricing kernel is a strictly positive Ito process driven by a Brownian motion $\{W_t\}$ and satisfies a dynamical equation of the form

$$\mathrm{d}\pi_t = \alpha_t\, \pi_t\, \mathrm{d}t + \beta_t\, \pi_t\, \mathrm{d}W_t\,. \tag{16}$$

Let $\{S_t\}$ be the price of a non-dividend-paying asset driven by the same Brownian motion. Then $\pi_t S_t = M_t$ for some positive martingale $\{M_t\}$ driven by $\{W_t\}$. Let the dynamics of $\{M_t\}$ be given by $\mathrm{d}M_t = \nu_t M_t\, \mathrm{d}W_t$. A calculation using Ito's formula then shows that

$$\mathrm{d}S_t = [-\alpha_t - \beta_t(\nu_t - \beta_t)]\, S_t\, \mathrm{d}t + (\nu_t - \beta_t)\, S_t\, \mathrm{d}W_t\,. \tag{17}$$

Thus if one writes $r_t = -\alpha_t$, $\lambda_t = -\beta_t$, and $\sigma_t = \nu_t - \beta_t$, it follows that the dynamics of the asset price takes the familiar form

$$\mathrm{d}S_t = (r_t + \lambda_t\sigma_t)\, S_t\, \mathrm{d}t + \sigma_t\, S_t\, \mathrm{d}W_t\,. \tag{18}$$

We see that r_t is the short rate of interest, that λ_t is the market price of risk, that σ_t is the volatility of the asset, and that the pricing kernel satisfies

$$\mathrm{d}\pi_t = -r_t\pi_t\, \mathrm{d}t - \lambda_t\pi_t\, \mathrm{d}W_t\,. \tag{19}$$

Combining (14) and (19), we deduce that in the Bessel(3) model for the pricing kernel the short rate satisfies $r_t = 0$ for all $t \geq 0$, and the market price of risk is given by $\lambda_t = \sigma_t\pi_t$. The vanishing of the short rate does not, however, imply the vanishing of rates of finite tenor, such as Libor rates and swap rates, as we shall see below.

3. Discount bonds and yields.

To work out the bond price process we shall be using the pricing formula (5):

$$P_{tT} = \frac{1}{\pi_t}\mathbb{E}_t\left[\frac{1}{\sqrt{(X_T - a)^2 + (Y_T - b)^2 + (Z_T - c)^2}}\right], \tag{20}$$

where $\mathbb{E}_t[\,\cdot\,] := \mathbb{E}[\,\cdot\,|\,\mathcal{F}_t]$. Writing $X_T - a = (X_T - X_t) + (X_t - a)$, and similarly for $Y_T - b$ and $Z_T - c$, we observe that the increments $X_T - X_t$, $Y_T - Y_t$, and $Z_T - Z_t$ are independent of $\mathcal{F}_t$, whereas $X_t - a$, $Y_t - b$, and $Z_t - c$ are $\mathcal{F}_t$-measurable. Thus, defining $X = X_T - a$, $Y = Y_T - b$, and $Z = Z_T - c$, then conditionally on $\mathcal{F}_t$ we have $X \sim N(X_t - a, \Sigma_{tT})$, $Y \sim N(Y_t - b, \Sigma_{tT})$ and $Z \sim N(Z_t - c, \Sigma_{tT})$, where

$$\Sigma_{tT} = \int_t^T \sigma_s^2\, \mathrm{d}s \tag{21}$$

is the conditional variance of the random variables X, Y, and Z. If we define the vectors $\boldsymbol{R} = (x, y, z)$ and $\boldsymbol{\xi}_t = (X_t - a, Y_t - b, Z_t - c)$, and their squared norms $R^2 = \boldsymbol{R}\cdot\boldsymbol{R}$ and $\xi_t^2 = \boldsymbol{\xi}_t \cdot \boldsymbol{\xi}_t$, then the bond price is given by

$$P_{tT} = \frac{1}{\pi_t}\frac{1}{(\sqrt{2\pi\Sigma_{tT}}\,)^3}\int_{\mathbb{R}^3}\frac{1}{R}\,\mathrm{e}^{-\frac{1}{2}\Sigma_{tT}^{-1}\,|\boldsymbol{R}-\boldsymbol{\xi}_t|^2}\mathrm{d}^3\boldsymbol{R}\,. \tag{22}$$

Thus using spherical representation

$$\mathrm{d}^3\boldsymbol{R} = R^2 \sin\theta \, \mathrm{d}R \, \mathrm{d}\theta \, \mathrm{d}\phi \tag{23}$$

for the volume element in $\mathbb{R}^3$ we deduce that

$$\begin{aligned}
P_{tT} &= \frac{1}{\pi_t} \frac{1}{\sqrt{2\pi}\Sigma_{tT}^{3/2}} \int_0^\infty R^2 \, \frac{1}{R} \int_0^\pi \sin\theta \, \mathrm{e}^{-\frac{1}{2}\Sigma_{tT}^{-1}(R^2 - 2R\xi_t\cos\theta + \xi_t^2)} \mathrm{d}\theta \, \mathrm{d}R \\
&= \frac{1}{\pi_t} \frac{1}{\sqrt{2\pi}\Sigma_{tT}^{3/2}} \, \mathrm{e}^{-\frac{1}{2}\Sigma_{tT}^{-1}\xi_t^2} \int_0^\infty R \, \mathrm{e}^{-\frac{1}{2}\Sigma_{tT}^{-1}R^2} \int_0^\pi \sin\theta \, \mathrm{e}^{R\xi_t\cos\theta/\Sigma_{tT}} \mathrm{d}\theta \, \mathrm{d}R \\
&= \frac{1}{\pi_t} \frac{1}{\sqrt{2\pi}\Sigma_{tT}^{3/2}} \, \mathrm{e}^{-\frac{1}{2}\Sigma_{tT}^{-1}\xi_t^2} \int_0^\infty R \, \mathrm{e}^{-\frac{1}{2}\Sigma_{tT}^{-1}R^2} \left[\frac{-\mathrm{e}^{R\xi_t\cos\theta/\Sigma_{tT}}}{R\xi_t/\Sigma_{tT}} \bigg|_0^\pi \right] \mathrm{d}R \\
&= \frac{1}{\sqrt{2\pi\Sigma_{tT}}} \, \mathrm{e}^{-\frac{1}{2}\Sigma_{tT}^{-1}\xi_t^2} \int_0^\infty \mathrm{e}^{-\frac{1}{2}\Sigma_{tT}^{-1}R^2} \left(\mathrm{e}^{R\xi_t/\Sigma_{tT}} - \mathrm{e}^{-R\xi_t/\Sigma_{tT}} \right) \mathrm{d}R \, ,
\end{aligned} \tag{24}$$

where we have made use of the fact that $\xi_t = 1/\pi_t$. We note that the process $\{\xi_t\}$ appearing here is the price of the so-called natural numeraire or benchmark asset [18, 21]. Completing the squares in the exponents in (24), we obtain

$$\begin{aligned}
P_{tT} &= \frac{1}{\sqrt{2\pi\Sigma_{tT}}} \int_0^\infty \left(\mathrm{e}^{-\frac{1}{2}\Sigma_{tT}^{-1}(R-\xi_t)^2} - \mathrm{e}^{-\frac{1}{2}\Sigma_{tT}^{-1}(R+\xi_t)^2} \right) \mathrm{d}R \\
&= \frac{1}{\sqrt{\pi}} \int_{-\xi_t/\sqrt{2\Sigma_{tT}}}^{\xi_t/\sqrt{2\Sigma_{tT}}} \mathrm{e}^{-u^2} \mathrm{d}u \, .
\end{aligned} \tag{25}$$

Let us now define the error function as usual by setting

$$\mathrm{erf}(z) = \frac{1}{\sqrt{\pi}} \int_{-z}^{z} \mathrm{e}^{-u^2} \mathrm{d}u = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} \, z^{2n+1}, \tag{26}$$

which is an entire function on the complex plane. Thus, $\mathrm{erf}(z) = N(\sqrt{2}\,z) - N(-\sqrt{2}\,z)$, where $N(x)$ is the normal distribution function. Then we obtain the following expression for the bond price:

$$P_{tT} = \mathrm{erf}\left(\sqrt{\frac{(X_t - a)^2 + (Y_t - b)^2 + (Z_t - c)^2}{2\,\Sigma_{tT}}} \right). \tag{27}$$

Equivalently, we have

$$P_{tT} = \mathrm{erf}\left(\frac{\xi_t}{\sqrt{2\Sigma_{tT}}} \right), \tag{28}$$

where $\{\xi_t\}$ is the natural numeraire. For illustration, we have shown in Figure 1 some sample paths for the bond price. As the price process of an asset, the discount bond satisfies the

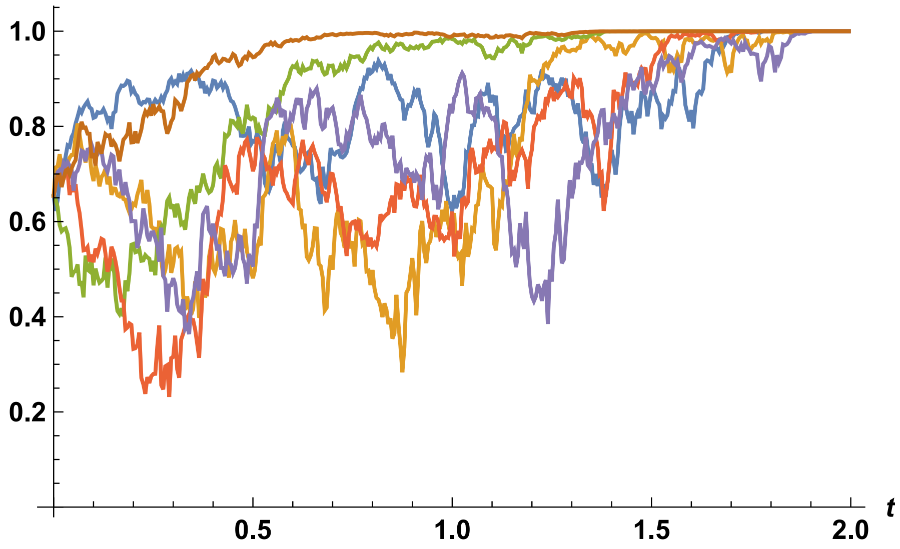

Figure 1. *The discount bond price process* $\{P_{tT}\}$ *with the choice* $\sigma_t = 0.75$ *and* $T = 2$*. Six sample paths are displayed, illustrating the qualitative behavior of the bond price process in the Bessel*(3) *model when the volatility function* $\{\sigma_t\}$ *is constant.*

condition that $\{\pi_t P_{tT}\}$ is a martingale, which follows at once from the tower property of conditional expectation:

$$\mathbb{E}_s[\pi_t P_{tT}] = \mathbb{E}_s[\mathbb{E}_t[\pi_T]] = \mathbb{E}_s[\pi_T] = \pi_s P_{sT}\,. \tag{29}$$

Alternatively, the martingale condition can be checked directly from expression (27) for the bond price if we make use of the identity

$$\frac{1}{\sqrt{\pi\alpha}}\int_0^\infty \left(\mathrm{e}^{-\frac{1}{\alpha}(\xi-x)^2} - \mathrm{e}^{-\frac{1}{\alpha}(\xi+x)^2}\right)\mathrm{erf}\left(\frac{\xi}{\sqrt{\beta}}\right)\mathrm{d}\xi = \mathrm{erf}\left(\frac{x}{\sqrt{\alpha+\beta}}\right). \tag{30}$$

We have seen that the model entails no interest since the drift of the pricing kernel is identically zero. Yet, bond prices give rise to discounting; in other words, P_{tT} is a decreasing function of T for each $t < T$, since Σ_{tT} is an increasing of T for $t < T$. One might therefore wonder whether the short rate vanishes if one employs the alternative definition of the short rate given by

$$r_t = -\left.\frac{\partial P_{tT}}{\partial T}\right|_{T=t}. \tag{31}$$

This can easily be checked. We have

$$r_t = \left.\frac{\sigma_T^2}{\sqrt{2\pi}\Sigma_{tT}^{3/2}}\exp\left(-\frac{\xi_t^2}{2\Sigma_{tT}}\right)\right|_{T=t}. \tag{32}$$

Since $\lim_{t\to T}\Sigma_{tT} = 0$, the exponential term suppresses the right side to give $r_t = 0$ for all $t \geq 0$. Alternatively, by use of (27), a calculation shows that

$$\mathrm{d}P_{tT} = \lambda_t \Omega_{tT} P_{tT}\,\mathrm{d}t + \Omega_{tT} P_{tT}\,\mathrm{d}W_t, \tag{33}$$

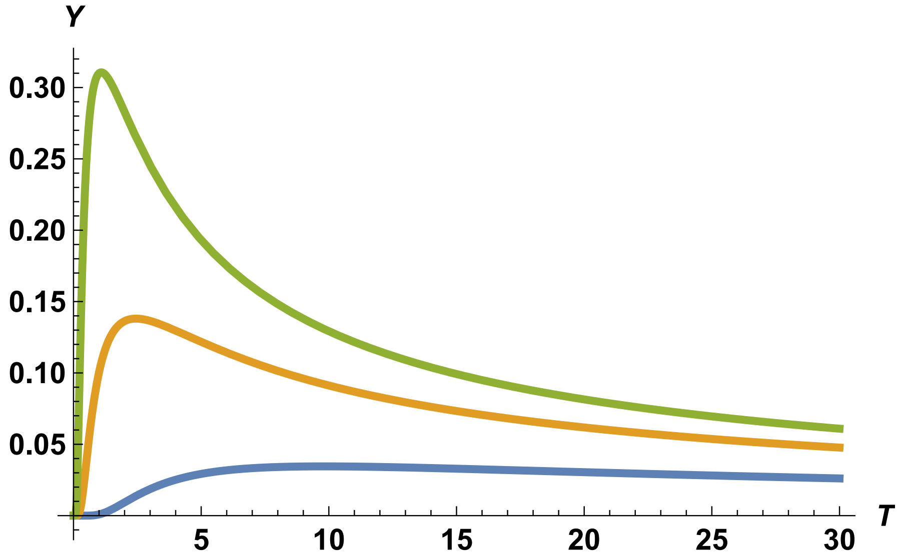

Figure 2. *The initial yield curve $Y(T)_{T\geq 0}$ for constant σ_t. Since $r_0 = 0$, the yield at $T = 0$ vanishes. With $\sigma_t = \sigma$, for constant σ, the yield curve peaks and then decays to zero, where $Y(T) \sim -\log(\sqrt{2/\pi\sigma T})/T$ as $T \to \infty$. We have plotted $\{Y(T)\}$ for $\sigma = 0.3$ (blue), $\sigma = 0.6$ (orange), and $\sigma = 0.9$ (green).*

where $\lambda_t = \sigma_t \pi_t$ is the market price of risk, and where

$$\Omega_{tT} = \frac{2\sigma_t}{P_{tT}\sqrt{2\pi\Sigma_{tT}}} \exp\left(-\frac{1}{2\Sigma_{tT}\pi_t^2}\right) \tag{34}$$

is the discount bond volatility. We note that $\lim_{t\to T} \Omega_{tT} = 0$. The form of (33) confirms that the contribution $r_t P_{tT}$ normally arising from the short rate in the drift is absent.

But the fact that the short rate is zero does not imply that other rates are necessarily zero. For instance, as a consequence of the definition

$$f_{tT} = -\frac{\partial \log P_{tT}}{\partial T}, \tag{35}$$

a calculation shows that the instantaneous forward rates are given by

$$f_{tT} = \frac{\sigma_T^2 \;\exp\left(-\xi_t^2/2\Sigma_{tT}\right)}{\sqrt{2\pi}\Sigma_{tT}^{3/2} \;\mathrm{erf}\left(\xi_t/\sqrt{2\Sigma_{tT}}\right)}, \tag{36}$$

and we see that $\lim_{t\to T} f_{tT} = 0$. Similarly, for the yield curve $\{Y(T)\}$ we obtain the following:

$$Y(T) = -\frac{1}{T}\log\left[\mathrm{erf}\left(\frac{1}{\sqrt{2\Sigma_{0T}}}\right)\right]. \tag{37}$$

Thus, initial yield curve data can be used to calibrate the freedom in the function $\{\sigma_t\}$. Our calibration scheme is essentially equivalent to that suggested in [10] using a time-change technique. A typical set of yield curves arising from constant $\{\sigma_t\}$ is sketched in Figure 2.

4. Bond options. Let us consider the pricing of options on discount bonds. To begin, we look at a European-style digital call option with maturity t and strike K on a discount bond that matures at time T. Thus, the option delivers one unit of cryptocurrency at time t in the event that $P_{tT} > K$. The option payout is the indicator function

$$H_t = \mathbb{1}\{P_{tT} > K\}, \tag{38}$$

so the price of a digital call is given by

$$D_0 = \mathbb{E}\left[\frac{1}{\xi_t}\mathbb{1}\left\{\operatorname{erf}\left(\frac{\xi_t}{\sqrt{2\Sigma_{tT}}}\right) > K\right\}\right], \tag{39}$$

where $\xi_t = \pi_t^{-1}$. The error function is increasing in its argument, so we find that there is a critical value ξ^* of the natural numeraire such that the option expires in the money if $\xi_t > \xi^*$, given by

$$\xi^* = \sqrt{2\Sigma_{tT}}\,\operatorname{erf}^{-1}(K). \tag{40}$$

Therefore, if we switch to a spherical representation for the volume element in $\mathbb{R}^3$, a calculation similar to that presented in (24) shows that the price of a digital call is

$$D_0 = \frac{1}{2}\left[\operatorname{erf}\left(\frac{\xi^*+1}{\sqrt{2\Sigma_{0t}}}\right) - \operatorname{erf}\left(\frac{\xi^*-1}{\sqrt{2\Sigma_{0t}}}\right)\right]. \tag{41}$$

More generally, let us consider the price process $\{D_s\}$ of the digital call option, given by the following expression:

$$D_s = \mathbb{1}_{\{0\le s<t\}}\frac{1}{\pi_s}\mathbb{E}_s[\pi_t H_t]. \tag{42}$$

Noticing that conditional on $\mathcal{F}_s$ we have $X_t \sim N(X_s - a, \Sigma_{st})$, $Y_t \sim N(Y_s - b, \Sigma_{st})$, and $Z_t \sim N(Z_s - c, \Sigma_{st})$, one finds that a calculation analogous to that considered in (24) leads to the formula

$$D_s = \frac{1}{2\xi_s}\left[\operatorname{erf}\left(\frac{\xi^*+\xi_s}{\sqrt{2\Sigma_{st}}}\right) - \operatorname{erf}\left(\frac{\xi^*-\xi_s}{\sqrt{2\Sigma_{st}}}\right)\right]. \tag{43}$$

We turn now to look at the pricing of an in-arrears caplet, for which the payout at time T is given by

$$H_T = X(L_{tT} - R)^+, \tag{44}$$

where R is the cap and X is the notional. The crypto rate L_{tT} appearing here is defined by

$$L_{tT} = \frac{1}{T-t}\left(\frac{1}{P_{tT}} - 1\right). \tag{45}$$

Since the caplet is paid "in arrears," meaning that the payoff is set at the earlier time t and paid at T, and since L_{tT} is known at time t, we can regard the caplet as a derivative that effectively pays the discounted value $H_t = P_{tT} H_T$ at the earlier time t. By substitution and rearrangement one sees that the effective payout at time t takes the form $H_t = N\,(K - P_{tT})^+$, where K and N are given by

$$K = \frac{1}{1+R(T-t)} \quad \text{and} \quad N = \frac{X[1+R(T-t)]}{T-t}\,. \tag{46}$$

Thus we see that a position in an in-arrears caplet is equivalent to a position in N puts on a discount bond, where the strike K on the put is the value of a discount bond with simple crypto yield R. Making use of (4) we deduce that the price of the caplet is

$$\begin{aligned} C_0 &= N\,\mathbb{E}\left[\pi_t\,(K - P_{tT})^+\right] \\ &= N\,\mathbb{E}\left[\frac{1}{\xi_t}\left(K - \mathrm{erf}\left(\frac{\xi_t}{\sqrt{2\Sigma_{tT}}}\right)\right)^+\right]. \end{aligned} \tag{47}$$

If we switch to the spherical representation for the volume element in $\mathbb{R}^3$, a calculation analogous to that presented in (24) shows that the option price can be represented in terms of the following Gaussian integrals:

$$\begin{aligned} C_0 &= \frac{1}{\sqrt{2\pi\Sigma_{0t}}}\int_0^{\xi^*}\left[K - \mathrm{erf}\left(\frac{R}{\sqrt{2\Sigma_{tT}}}\right)\right]\left(\mathrm{e}^{-\frac{1}{2\Sigma_{0t}}(R-1)^2} - \mathrm{e}^{-\frac{1}{2\Sigma_{0t}}(R+1)^2}\right)\mathrm{d}R \\ &= \frac{1}{\sqrt{\pi}}\int_{\frac{1}{\sqrt{2\Sigma_{0t}}}}^{\frac{\xi^*+1}{\sqrt{2\Sigma_{tT}}}} \mathrm{e}^{-u^2}\mathrm{erf}\left(\tfrac{\sqrt{2\Sigma_{0t}}\,u-1}{\sqrt{2\Sigma_{tT}}}\right)\mathrm{d}u - \frac{1}{\sqrt{\pi}}\int_{\frac{-1}{\sqrt{2\Sigma_{0t}}}}^{\frac{\xi^*-1}{\sqrt{2\Sigma_{tT}}}} \mathrm{e}^{-u^2}\mathrm{erf}\left(\tfrac{\sqrt{2\Sigma_{0t}}\,u+1}{\sqrt{2\Sigma_{tT}}}\right)\mathrm{d}u \\ &\quad + K\left[\mathrm{erf}\left(\tfrac{1}{\sqrt{2\Sigma_{0t}}}\right) - \tfrac{1}{2}\left(\mathrm{erf}\left(\tfrac{\xi^*+1}{\sqrt{2\Sigma_{0t}}}\right) - \mathrm{erf}\left(\tfrac{\xi^*-1}{\sqrt{2\Sigma_{0t}}}\right)\right)\right]. \end{aligned} \tag{48}$$

While there appears to be no simpler representation for the Gaussian integrals appearing here, numerical evaluation is straightforward.

5. Models based on higher-order Bessel processes.

Bessel processes of order four or more also give rise to cryptobond models, analogous to the one we have already investigated. In particular, if we consider a collection of Gaussian processes $\{X^a_t\}$ for $a = 1, 2, \ldots, n$ of the type given by (10) in dimension $n \geq 3$, then we can model the pricing kernel by setting

$$\pi_t = \left[(X^1_t)^2 + \cdots + (X^n_t)^2\right]^{(2-n)/2}. \tag{49}$$

A short calculation shows that

$$\mathrm{d}\pi_t = -(n-2)\,\sigma_t\,\pi_t^{(n-1)/(n-2)}\mathrm{d}W_t\,, \tag{50}$$

from which it follows on account of the test discussed in [19, 20] that $\{\pi_t\}$ is a strict local martingale for all $n \geq 3$.

As an illustration, we present a pricing kernel model for cryptocurrencies based on the reciprocal of the Bessel process in four dimensions. See, for example, reference [21] for properties of the Bessel(4) process. For our model we take

$$\pi_t = \frac{1}{(X^1_t - a)^2 + (X^2_t - b)^2 + (X^3_t - c)^2 + (X^4_t - d)^2}\,, \tag{51}$$

where the $\{X_t^k\}_{k=1,\ldots,4}$ are four independent Gaussian processes of the form (10), and the constants a, b, c, d are chosen such that $a^2 + b^2 + c^2 + d^2 = 1$. A calculation shows that the dynamical equation of the pricing kernel is

$$\mathrm{d}\pi_t = -2\sigma_t \, \pi_t^{3/2} \mathrm{d}W_t \,, \tag{52}$$

which corresponds to (50) for $n = 4$. We wish to compute the discount bond price in this model, which, following the logic of (22), with $\boldsymbol{\xi}_t = (X_t^1 - a, X_t^2 - b, X_t^3 - c, X_t^4 - d)$, is given by

$$P_{tT} = \frac{1}{\pi_t} \frac{1}{(\sqrt{2\pi\Sigma_{tT}})^4} \int_{\mathbb{R}^4} \frac{1}{R^2} \, \mathrm{e}^{-\frac{1}{2}\Sigma_{tT}^{-1} \, |\boldsymbol{R} - \boldsymbol{\xi}_t|^2} \mathrm{d}^4 \boldsymbol{R} \,. \tag{53}$$

We switch to a spherical representation. In four dimensions, we set $x = R \sin\theta \, \sin\varphi \, \cos\phi$, $y = R \sin\theta \, \sin\varphi \, \sin\phi$, $z = R \sin\theta \, \cos\varphi$, and $w = R\cos\theta$, with the volume element

$$\mathrm{d}^4 \boldsymbol{R} = R^3 \sin^2\theta \, \sin\varphi \, \mathrm{d}R \, \mathrm{d}\theta \, \mathrm{d}\varphi \, \mathrm{d}\phi \,. \tag{54}$$

Note that $\theta, \varphi \in [0, \pi]$ and $\phi \in [0, 2\pi]$. Since the vector $\boldsymbol{\xi}_t$ is fixed, and because of the spherical symmetry, we may without loss of generality choose $\boldsymbol{\xi}_t$ to be in the direction of the w-axis. A similar assumption was made in the three-dimensional case in (24), where $\boldsymbol{\xi}_t$ was taken to be in the z-direction. Then we have $\boldsymbol{R} \cdot \boldsymbol{\xi}_t = R\xi_t \cos\theta$, a choice that simplifies the calculation somewhat. Integration over ϕ gives 2π, whereas $\int_0^\pi \sin\varphi \, \mathrm{d}\varphi = 2$, so after performing the integration over these variables we obtain

$$P_{tT} = \frac{1}{\pi_t} \frac{2}{2\pi\Sigma_{tT}^2} \int_0^\infty R \int_0^\pi \sin^2\theta \, \mathrm{e}^{-\frac{1}{2}\Sigma_{tT}^{-1} \, (R^2 - 2R\xi_t \cos\theta + \xi_t^2)} \mathrm{d}\theta \, \mathrm{d}R \,, \tag{55}$$

where $\{\xi_t\}$ represents the Bessel process in four dimensions, so $\pi_t = \xi_t^{-2}$. To proceed, we note the identity

$$\int_0^\pi \sin^2\theta \, \mathrm{e}^{\nu\cos\theta} \mathrm{d}\theta = \frac{\pi}{\nu} \, I_1(\nu). \tag{56}$$

This follows if we observe that $\sin^2\theta \, \mathrm{e}^{\nu\cos\theta} = (\sin\theta)(\sin\theta \, \mathrm{e}^{\nu\cos\theta})$ and that $\sin\theta \, \mathrm{e}^{\nu\cos\theta} = -\nu^{-1}\partial_\theta \, \mathrm{e}^{\nu\cos\theta}$, which shows that we can carry out an integration by parts to reduce the integrand to $\cos\theta \, \mathrm{e}^{\nu\cos\theta}$. But then we notice that $\cos\theta \, \mathrm{e}^{\nu\cos\theta} = \partial_\nu \, \mathrm{e}^{\nu\cos\theta}$, so moving ∂_ν outside the integration we see that the integrand reduces further to $\mathrm{e}^{\nu\cos\theta}$. But this gives rise to a Bessel function, and we have $\int_0^\pi \mathrm{e}^{\nu\cos\theta} \mathrm{d}\theta = \pi \, I_0(\nu)$. Differentiating and using the differential identity $\partial_\nu \, I_0(\nu) = I_1(\nu)$, we arrive at the conclusion. Alternatively, if we recall the definition

$$I_n(\nu) = \frac{1}{\pi} \int_0^\pi \mathrm{e}^{\nu\cos\theta} \cos(n\theta) \mathrm{d}\theta \tag{57}$$

for the generalized Bessel function of the first kind, we arrive at the same conclusion more expediently. In any case, we deduce that

$$\int_0^\pi \sin^2\theta \, \mathrm{e}^{R\,\xi_t \, \Sigma_{tT}^{-1} \, \cos\theta} \mathrm{d}\theta = \frac{\pi\Sigma_{tT}}{R\xi_t} \, I_1\!\left(\tfrac{R\xi_t}{\Sigma_{tT}}\right). \tag{58}$$

Thus using $(\pi_t \xi_t)^{-1} = \xi_t$ we obtain

$$P_{tT} = \frac{\xi_t}{\Sigma_{tT}} \int_0^\infty \mathrm{e}^{-\frac{1}{2}\Sigma_{tT}^{-1}(R^2+\xi_t^2)} I_1\left(\tfrac{R\xi_t}{\Sigma_{tT}}\right) \mathrm{d}R. \tag{59}$$

If we define $u = R/\sqrt{\Sigma_{tT}}$ and $\eta_t = \xi_t/\sqrt{\Sigma_{tT}}$, the expression simplifies to

$$P_{tT} = \eta_t\, \mathrm{e}^{-\frac{1}{2}\eta_t^2} \int_0^\infty \mathrm{e}^{-\frac{1}{2}u^2} I_1(\eta_t u)\, \mathrm{d}u. \tag{60}$$

Now we use the identity

$$\int_0^\infty \mathrm{e}^{-\frac{1}{2}u^2} I_1(\eta u)\, \mathrm{d}u = \frac{\mathrm{e}^{\frac{1}{2}\eta^2} - 1}{\eta}, \tag{61}$$

which can be established by use of the Taylor series expansion of the Bessel function

$$I_n(\nu) = \sum_{k=0}^\infty \frac{1}{2^{2k+n} k!\,\Gamma(n+k+1)}\, \nu^{2k+n} \tag{62}$$

along with the expression

$$\int_0^\infty \mathrm{e}^{-\frac{1}{2}u^2} u^{2k+n} \mathrm{d}u = 2^{(2k+n-1)/2}\, \Gamma\left(\frac{2k+n+1}{2}\right) \tag{63}$$

for the Gaussian moments. Specifically, substituting (62) for $n = 1$ and $\nu = \eta u$ in the left side of (61) and using (63) for $n = 1$, we obtain

$$\begin{aligned} \int_0^\infty \mathrm{e}^{-\frac{1}{2}u^2} I_1(\eta u)\, \mathrm{d}u &= \sum_{k=0}^\infty \frac{\eta^{2k+1}}{2^{2k+1} k!\,\Gamma(k+2)}\, 2^k \Gamma(k+1) \\ &= \sum_{k=0}^\infty \frac{(\eta/2)^{k+1}}{(k+1)!} \\ &= \frac{1}{\eta}\left(\sum_{k=0}^\infty \frac{(\eta/2)^k}{k!} - 1\right), \end{aligned} \tag{64}$$

and this establishes (61). Putting these together we arrive at the bond price

$$P_{tT} = 1 - \mathrm{e}^{-\frac{1}{2}\Sigma_{tT}^{-1}\xi_t^2}, \tag{65}$$

which turns out to be surprisingly simple.

For illustration we have shown in Figure 3 some sample paths for the bond price process. Note that $\lim_{t\to T} P_{tT} = 1$, whereas, assuming that $\sigma_t > 0$ for all $t \geq 0$, we have $\lim_{T\to\infty} \Sigma_{tT} = \infty$, from which it follows that $\lim_{T\to\infty} P_{tT} \to 0$. The initial bond price is given by

$$P_{0T} = 1 - \exp\left(-\frac{1}{2\Sigma_{0T}}\right), \tag{66}$$

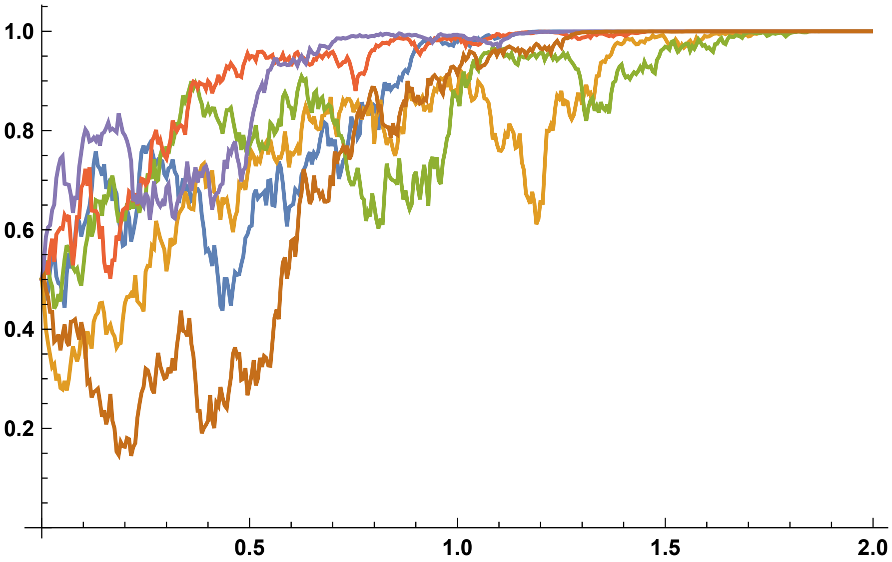

Figure 3. *The discount bond price process with $\sigma_t = 0.6$ and $T = 2$. Six sample paths are displayed, illustrating the qualitative behavior of the bond price in a model based on the Bessel(4) process.*

from which we deduce that the initial yield curve takes the form

$$Y(T) = -\frac{1}{T}\log\left[1 - \exp\left(-\frac{1}{2\Sigma_{0T}}\right)\right]. \tag{67}$$

This relation can be used to calibrate the volatility function to market data. Specifically, we have

$$\sigma_T^2 = -\frac{(Y(T) + TY'(T))\,\mathrm{e}^{-TY(T)}}{2(1 - \mathrm{e}^{-TY(T)})\left(\log(1 - \mathrm{e}^{-TY(T)})\right)^2}, \tag{68}$$

which allows us to determine the form of the function $\{\sigma_t\}_{t\geq 0}$ from any initial yield curve $\{Y(t)\}_{t\geq 0}$ satisfying the constraint $Y(0) = 0$.

Next we examine the dynamics of the bond price. If we start with

$$\mathrm{d}\xi_t = \frac{3\sigma_t^2}{2\xi_t}\,\mathrm{d}t + \sigma_t\,\mathrm{d}W_t, \tag{69}$$

an application of Ito's formula gives

$$\mathrm{d}P_{tT} = \lambda_t\Omega_{tT}P_{tT}\,\mathrm{d}t + \Omega_{tT}P_{tT}\,\mathrm{d}W_t, \tag{70}$$

where

$$\lambda_t = 2\sigma_t\xi_t^{-1} \qquad \text{and} \qquad \Omega_{tT} = \frac{\sigma_t\xi_t}{P_{tT}\Sigma_{tT}}\,\mathrm{e}^{-\frac{1}{2}\Sigma_{tT}^{-1}\xi_t^2}. \tag{71}$$

This result offers an independent confirmation of the fact that $r_t = 0$ for all $t \geq 0$ in the present model.

Let us now consider the valuation of a call option on a discount bond. The payoff takes the form $H_t = (P_{tT} - K)^+$, where K is the strike price of the option, t is the expiration date of the option, and $T > t$ is the maturity date of the bond. We assume that $0 < K < 1$. Since the bond price is an increasing function of ξ_t, we find that there is a critical value

$$\xi^* = \sqrt{-2\Sigma_{tT} \log(1-K)} \tag{72}$$

such that $H_t = 0$ if $\xi_t \leq \xi^*$. After we perform the integration over the (ϕ, φ) variables, we find that the initial price of the option is determined by the integral

$$C_0 = \frac{1}{\pi \Sigma_{0t}^2} \int_{\xi^*}^{\infty} R \left((1-K) - \mathrm{e}^{-\frac{1}{2\Sigma_{tT}} R^2} \right) \mathrm{e}^{-\frac{1}{2\Sigma_{0t}}(R^2+1)} \int_0^{\pi} \sin^2\theta \, \mathrm{e}^{\frac{R}{\Sigma_{0t}}\cos\theta} \mathrm{d}\theta \, \mathrm{d}R \,. \tag{73}$$

Performing the θ integration, we thus have

$$C_0 = \frac{1}{\Sigma_{0t}} \int_{\xi^*}^{\infty} \left((1-K) - \mathrm{e}^{-\frac{1}{2\Sigma_{tT}} R^2} \right) I_1\left(\frac{R}{\Sigma_{0t}} \right) \mathrm{e}^{-\frac{1}{2\Sigma_{0t}}(R^2+1)} \, \mathrm{d}R \,. \tag{74}$$

Similarly, for a put option with payout $(K - P_{tT})^+$, we obtain

$$P_0 = \frac{1}{\Sigma_{0t}} \int_0^{\xi^*} \left((K-1) + \mathrm{e}^{-\frac{1}{2\Sigma_{tT}} R^2} \right) I_1\left(\frac{R}{\Sigma_{0t}} \right) \mathrm{e}^{-\frac{1}{2\Sigma_{0t}}(R^2+1)} \, \mathrm{d}R \,, \tag{75}$$

from which we observe that

$$C_0 - P_0 = \frac{1}{\Sigma_{0t}} \int_0^{\infty} \left((K-1) + \mathrm{e}^{-\frac{1}{2\Sigma_{tT}} R^2} \right) I_1\left(\frac{R}{\Sigma_{0t}} \right) \mathrm{e}^{-\frac{1}{2\Sigma_{0t}}(R^2+1)} \, \mathrm{d}R \,. \tag{76}$$

Using (61) we can integrate the right side of (76) explicitly to obtain the put-call parity relation:

$$\begin{aligned} C_0 - P_0 &= (1-K)\left(1 - \mathrm{e}^{-\frac{1}{2\Sigma_{0t}}} \right) + \mathrm{e}^{-\frac{1}{2\Sigma_{0t}}} - \mathrm{e}^{-\frac{1}{2\Sigma_{0t}}\left[1 - \frac{\Sigma_{tT}}{\Sigma_{0T}}\right]} \\ &= P_{0T} - K P_{0t}, \end{aligned} \tag{77}$$

where we have made use of the fact that $\Sigma_{0t} + \Sigma_{tT} = \Sigma_{0T}$.

The indefinite Gaussian integrals of the Bessel function for the option prices have to be evaluated numerically. It is interesting to note that despite the simplicity in the model for the bond price, the option price cannot be expressed in closed form in terms of known functions. Nevertheless, fast numerical valuation is straightforward. To see this, we use the Taylor series expansion (62) for the Bessel function to obtain

$$C_0 = \frac{1}{\Sigma_{0t}} \mathrm{e}^{-\frac{1}{2\Sigma_{0t}}} \sum_{k=0}^{\infty} \frac{(1/2\Sigma_{0t})^{2k+1}}{k!(k+1)!} \int_{\xi^*}^{\infty} R^{2k+1} \left[(1-K) - \mathrm{e}^{-\frac{1}{2\Sigma_{tT}} R^2} \right] \mathrm{e}^{-\frac{1}{2\Sigma_{0t}} R^2} \mathrm{d}R \,. \tag{78}$$

Then, by changing the integration variable by setting $u = R^2$, we find that the integration reduces to that of an incomplete gamma function

$$\Gamma(a, z) = \int_z^{\infty} u^{a-1} \mathrm{e}^{-u} \mathrm{d}u, \tag{79}$$

and we thus obtain the option price in the form of a series:

$$
\begin{aligned}
C_0 = \mathrm{e}^{-\frac{1}{2\Sigma_{0t}}} \sum_{k=0}^{\infty} \frac{(1/2\Sigma_{0t})^{k+1}}{k!(k+1)!} \Bigg[&(1-K)\,\Gamma\left(k+1, -\frac{\Sigma_{tT}}{\Sigma_{0t}} \log(1-K)\right) \\
&- \left(\frac{\Sigma_{tT}}{\Sigma_{0T}}\right)^{k+1} \Gamma\left(k+1, -\frac{\Sigma_{0T}}{\Sigma_{0t}} \log(1-K)\right)\Bigg].
\end{aligned} \tag{80}
$$

On account of the appearance of the double factorial in the denominator in the summand in the expression above, the series converges rapidly, making it a useful expression for numerical valuation of the option price.

In particular, truncating the sum at, say, $k = 20$, we can obtain option prices very rapidly by use of standard numerical tools; the difference of the result thus obtained and the result of a standard numerical valuation of the integral (78) is of the order 10^{-16}.

More generally, in higher dimensions it should be evident that by use of the spherical representation for calculating the expectation $\mathbb{E}_t[\pi_T]$ it is always possible to set the direction of the vector $\boldsymbol{\xi}_t$ such that $\boldsymbol{R}\cdot\boldsymbol{\xi}_t = R\xi_t \cos\theta$. Thus the only nontrivial integration concerns the variables θ and R. Performing the θ integration we arrive at a linear combination of Bessel functions if the dimension n of the Bessel process is even, which then has to be integrated with respect to a Gaussian measure to obtain an expression for the bond price, whereas if n is odd, the θ integration gives rise to a linear combination of exponential functions, which again has to be integrated with respect to a Gaussian measure to obtain an expression for the bond price. Thus, depending on whether n is even or odd, for $n \geq 3$ we obtain two different types of cryptocurrency interest rate models.

6. Complex extensions of the model.

The model associated with the reciprocal of the Bessel process in three dimensions can be extended in an alternative manner to allow for parametric degrees of freedom to be incorporated. This can be achieved if we allow the parameters a, b, and c appearing in (11) to be complex numbers. The real part of the resulting complexified process $\{\pi_t^{\mathbb{C}}\}$ then defines an admissible model for the pricing kernel, with vanishing short rate. The reason for this is that when the parameters a, b, and c are complex, then both real and imaginary parts of the function $u(x, y, z)$ satisfy Laplace's equation, and the real part is strictly positive. The additional freedom thus arising can be used, for instance, to calibrate the model not only against the yield curve but also against option prices.

To proceed, let us therefore write $a = a_0 + \mathrm{i}a_1$, $b = b_0 + \mathrm{i}b_1$ and $c = c_0 + \mathrm{i}c_1$. Additionally, let us write $\tilde{x} = x - a_0$, $\tilde{y} = y - b_0$, and $\tilde{z} = z - c_0$. Then we have

$$
u(x,y,z) = \sqrt{\frac{\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2 - a_1^2 - b_1^2 - c_1^2 + 2\mathrm{i}(\tilde{x}a_1 + \tilde{y}b_1 + \tilde{z}c_1)}{\left(\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2 - a_1^2 - b_1^2 - c_1^2\right)^2 + 4\left(\tilde{x}a_1 + \tilde{y}b_1 + \tilde{z}c_1\right)^2}}\,. \tag{81}
$$

This function is a solution of Laplace's equation away from the "ring" singularity defined by the intersection of the two-sphere Σ of radius $\sqrt{a_1^2 + b_1^2 + c_1^2}$ centred at the point (a_0, b_0, c_0) and the two-plane Π defined by $a_1 x + b_1 y + c_1 z = a_0 a_1 + b_0 b_1 + c_0 c_1$, which passes through the point (a_0, b_0, c_0) and hence cuts Σ in an equatorial circle. We recall the formula

$$\sqrt{A+\mathrm{i}B} = \sqrt{\frac{A+\sqrt{A^2+B^2}}{2}} + \mathrm{i}\frac{B}{|B|}\sqrt{\frac{-A+\sqrt{A^2+B^2}}{2}} \tag{82}$$

for the real and the imaginary parts of the principal square-root of a complex number for which $B \neq 0$. It follows that $\mathrm{Re}\,(u) > 0$ on $\mathbb{R}^3\backslash\{\Sigma\cap\Pi\}$, whereas $\mathrm{Im}\,(u) = 0$ on $\Pi\backslash\{\Sigma\cap\Pi\}$. With these results at hand, we introduce a new crypto-rate model by setting

$$\pi_t = \mathrm{Re}\left(\pi_t^{\mathbb{C}}\right). \tag{83}$$

Then writing $\tilde{X}_t = X_t - a_0$, $\tilde{Y}_t = Y_t - b_0$, and $\tilde{Z}_t = Z_t - c_0$, we obtain the following expression for the pricing kernel:

$$\pi_t = \sqrt{\frac{\begin{matrix}\tilde{X}_t^2+\tilde{Y}_t^2+\tilde{Z}_t^2\\ -a_1^2-b_1^2-c_1^2\end{matrix} + \sqrt{\begin{matrix}\left(\tilde{X}_t^2+\tilde{Y}_t^2+\tilde{Z}_t^2-a_1^2-b_1^2-c_1^2\right)^2\\ +4\left(\tilde{X}_t a_1+\tilde{Y}_t b_1+\tilde{Z}_t c_1\right)^2\end{matrix}}}{2\left[\left(\tilde{X}_t^2+\tilde{Y}_t^2+\tilde{Z}_t^2-a_1^2-b_1^2-c_1^2\right)^2+4\left(\tilde{X}_t a_1+\tilde{Y}_t b_1+\tilde{Z}_t c_1\right)^2\right]}}. \tag{84}$$

The normalization $\pi_0 = 1$ imposes one constraint, whereas rotational symmetry can be used to eliminate two further parameters. Thus we are left with a model with three exogenously specifiable parameters that can be used to fit option prices.

To obtain an expression for the bond price, we need to work out the conditional expectation $\mathbb{E}_t[\pi_T]$. Rather than using expression (84) for the pricing kernel, which makes the computation somewhat cumbersome, we can take advantage of the fact that

$$\mathbb{E}_t\left[\mathrm{Re}\left(\pi_T^{\mathbb{C}}\right)\right] = \mathrm{Re}\left(\mathbb{E}_t\left[\pi_T^{\mathbb{C}}\right]\right). \tag{85}$$

Then we can use the simpler formula (11) for the pricing kernel, with complex parameters a, b, and c, and calculate its expectation, taking the real part of the result. Using the spherical representation, we have

$$\mathbb{E}_t[\pi_T^{\mathbb{C}}] = \frac{1}{(\sqrt{2\pi\Sigma_{tT}})^3}\int_{\mathbb{R}^3}\frac{1}{R}\,\mathrm{e}^{-\frac{1}{2\Sigma_{tT}}|\boldsymbol{R}-(\boldsymbol{\xi}_t-\mathrm{i}\boldsymbol{\delta})|^2}R^2\sin\theta\,\mathrm{d}R\,\mathrm{d}\theta\,\mathrm{d}\phi, \tag{86}$$

where $\boldsymbol{\xi}_t = (X_t - a_0, Y_t - b_0, Z_t - c_0)$ and $\boldsymbol{\delta} = (a_1, b_1, c_1)$.

It turns out that the calculation leading to (27) is applicable for complex parameters a, b, and c. To see this, we perform the integration explicitly. Recall that in the real case where $\boldsymbol{\delta} = \boldsymbol{0}$ we have one fixed vector $\boldsymbol{\xi}_t$ in the exponent of the integrand, so by using the spherical symmetry we choose this vector to point in the z direction, resulting in the simple expression $\boldsymbol{R}\cdot\boldsymbol{\xi}_t = R\xi_t\cos\theta$, which was used in the calculation of (27).

In the present case, we have two fixed vectors $\boldsymbol{\xi}_t$ and $\boldsymbol{\delta}$, so we can use the rotational symmetry to let the two vectors lie on the x-y plane, symmetrically placed about the x-axis. We let $2\alpha_t$ denote the angle between the two vectors $\boldsymbol{\xi}_t$ and $\boldsymbol{\delta}$. In other words, we have

$\boldsymbol{\xi}_t \cdot \boldsymbol{\delta} = \xi_t\, \delta \cos(2\alpha_t)$, where $\xi_t^2 = \boldsymbol{\xi}_t \cdot \boldsymbol{\xi}_t$ and $\delta^2 = \boldsymbol{\delta} \cdot \boldsymbol{\delta}$. Thus, the angle between $\boldsymbol{\xi}_t$ and the x-axis is α_t, and similarly the angle between $\boldsymbol{\delta}$ and the x-axis is $-\alpha_t$. With this choice of coordinates we have

$$\boldsymbol{R} \cdot (\boldsymbol{\xi}_t - \mathrm{i}\boldsymbol{\delta}) = \big(R(\xi_t - \mathrm{i}\delta) \sin\theta \cos\alpha_t\big) \cos\phi + \big(R(\xi_t + \mathrm{i}\delta) \sin\theta \sin\alpha_t\big) \sin\phi. \tag{87}$$

We are now in a position to perform the integration over the variable ϕ. To this end we recall the identity

$$\int_0^{2\pi} \mathrm{e}^{p\cos\phi + q\sin\phi} \mathrm{d}\phi = 2\pi I_0\left(\sqrt{p^2+q^2}\right). \tag{88}$$

This can be seen by viewing the exponent of the integrand as an inner product between the vector (p, q) and the unit vector placed at an angle ϕ from the vector (p, q). Then the exponent is equivalent to $\sqrt{p^2+q^2}\cos\phi$, and the result follows. In the present case we have $p = R(\xi_t - \mathrm{i}\delta)\sin\theta\cos\alpha_t$ and $q = R(\xi_t + \mathrm{i}\delta)\sin\theta\sin\alpha_t$, so $p^2 + q^2 = R^2\omega_t^2 \sin^2\theta$, where

$$\omega_t^2 = |(\boldsymbol{\xi}_t - \mathrm{i}\boldsymbol{\delta})|^2 = \xi_t^2 - \delta^2 - 2\mathrm{i}\xi_t\delta\cos(2\alpha_t). \tag{89}$$

We thus deduce that

$$\mathbb{E}_t\left[\pi_T^{\mathbb{C}}\right] = \frac{2\pi}{(\sqrt{2\pi\Sigma_{tT}})^3} \int_0^\infty \int_0^\pi R\, \mathrm{e}^{-\frac{1}{2\Sigma_{tT}}(R^2+\omega_t^2)} \sin\theta I_0\left(\frac{R\omega_t}{\Sigma_{tT}}\sin\theta\right) \mathrm{d}\theta \mathrm{d}R. \tag{90}$$

To perform the integration over the variable θ we note from (62) that

$$I_0(\nu\sin\theta) = \sum_{k=0}^\infty \frac{\nu^{2k}}{2^{2k}(k!)^2}(\sin\theta)^{2k}. \tag{91}$$

Thus, because

$$\int_0^\pi (\sin\theta)^{2k+1}\mathrm{d}\theta = \frac{2^{2k+1}(k!)^2}{(2k+1)!}, \tag{92}$$

and taking into account the Taylor series expansion

$$\frac{2\sinh(\nu)}{\nu} = 2\sum_{k=0}^\infty \frac{\nu^{2k}}{(2k+1)!}, \tag{93}$$

we deduce the identity

$$\int_0^\pi \sin\theta\, I_0(\nu\sin\theta)\mathrm{d}\theta = \frac{1}{\nu}\left(\mathrm{e}^{\nu} - \mathrm{e}^{-\nu}\right), \tag{94}$$

from which it follows that

$$\begin{aligned} \mathbb{E}_t\left[\pi_T^{\mathbb{C}}\right] &= \frac{\omega_t^{-1}}{\sqrt{2\pi\Sigma_{tT}}} \int_0^\infty \mathrm{e}^{-\frac{1}{2\Sigma_{tT}}(R^2+\omega_t^2)} \left(\mathrm{e}^{\frac{R\omega_t}{\Sigma_{tT}}} - \mathrm{e}^{-\frac{R\omega_t}{\Sigma_{tT}}}\right) \mathrm{d}R \\ &= \omega_t^{-1}\,\mathrm{erf}\left(\frac{\omega_t}{\sqrt{2\Sigma_{tT}}}\right). \end{aligned} \tag{95}$$

Noting that $\pi_t^{\mathbb{C}} = \omega_t^{-1}$ we thus validate the claim that the calculation leading to (27) is applicable for complex parameters a, b, and c. In particular, for the bond price we have

$$P_{tT} = \frac{1}{\mathrm{Re}(\omega_t^{-1})} \,\mathrm{Re}\left(\omega_t^{-1}\,\mathrm{erf}\left(\frac{\omega_t}{\sqrt{2\Sigma_{tT}}}\right)\right), \tag{96}$$

where ω_t is defined by (89). It should be apparent that in the real case for which $\boldsymbol{\delta} = 0$, we recover from (96) the previous expression (27) for the bond price.

Hence, by the complexification of models based on Bessel processes we can obtain genuine parametric extensions of the resulting term structure models. The complexification method that we have applied here is reminiscent of an analogous technique that has been used in physical applications [22, 23].

7. Discussion. The notion that strict local martingales should play a role in finance has been considered in various contexts by a number of authors. One can mention, in particular, the so-called benchmark approach of Platen and his collaborators and the Föllmer–Jarrow–Protter theory of price bubbles (see [21, 24, 25] and references cited therein) as examples that have attracted considerable attention. In the present paper, we have put forward an altogether different proposal for the application of local martingales in the theory of finance—namely, the idea that strict local martingales can be used as a basis for modeling the pricing kernel in a crypto economy where there is no money market account. The familiar rules of risk-neutral pricing no longer apply, since the money market account is not available to act as a numeraire. Nevertheless, as we have shown, the existence of a market price of risk is sufficient to ensure nontrivial discounting, despite the vanishing of the short rate. Our approach to crypto interest rates has been developed in some detail in models based on Bessel processes of order three and order four. These models have the advantage that explicit formulae, or semiexplicit expressions involving Gaussian integrals, can be obtained for the prices of a variety of derivative contracts.

More generally, one can envisage a market admitting numerous decentralized currencies. Now, in a friction-free market with n cryptocurrencies, if we write S_t^{ij} $(i, j = 1, 2, 3, \ldots, n)$ for the price at time t of one unit of currency i quoted in units of currency j, then we have

$$S_t^{ij} = \pi_t^i / \pi_t^j\,, \tag{97}$$

where $\{\pi_t^i\}$ denotes the pricing kernel for currency i [11, 18, 26]. In the present modeling framework, one can imagine a situation in which the pricing kernels each take the form (11), with initial values $\pi_0^i = (a_i^2 + b_i^2 + c_i^2)^{-1/2}$ and respective $\{\sigma_t^i\}$ functions. Some of the Brownian motions are shared throughout the crypto economy, representing systematic risk, while others may apply, perhaps, only to one or two cryptocurrencies, representing idiosyncratic risk. A call option on the crypto exchange rate with maturity T and strike rate K will have the payout $H_T = (\pi_T^i/\pi_T^j - K)^+$, the price of which can easily be computed numerically on account of the Gaussian nature of the setup.

In a similar vein, one can examine the problem of pricing a call option on the exchange rate between cryptocurrency i and a sovereign currency, say, USD. Then the dollar price of an option to purchase one unit of cryptocurrency i at time T at the strike price K is

$$C_0^{i\$} = \frac{1}{\pi_0^\$}\,\mathbb{E}\left[(\pi_T^i - K\pi_T^\$)^+\right]. \tag{98}$$

As an illustration, suppose that we have a geometric Brownian motion model

$$\pi_t^\$ = \pi_0^\$\,\mathrm{e}^{-rt-\lambda B_t-\frac{1}{2}\lambda^2 t} \tag{99}$$

for the dollar pricing kernel, where the short rate r is constant. For the cryptocurrency (say, bitcoin), we consider a pricing kernel of the form (11) with initial value $\pi_0^{฿} = (a^2+b^2+c^2)^{-1/2}$, and such that $\{B_t\}$ and $\{W_t\}$ are independent. The option price is then determined by the expectation of the random variable $(\pi_T^{฿} - K\pi_T^\$)^+$, which is nonzero only if

$$B_T > -\frac{1}{\lambda}\left[\log\left(\pi_T^{฿}/K\pi_0^\$\right) + rT + \frac{1}{2}\lambda^2 T\right]. \tag{100}$$

A straightforward calculation then shows that the option price is given by

$$C_0^{฿\$} = \frac{1}{\pi_0^\$}\,\mathbb{E}\left[\pi_T^{฿} N(g_+) - K\mathrm{e}^{-rT}\pi_0^\$ N(g_-)\right], \tag{101}$$

where

$$g_\pm = \frac{\log\left(\pi_T^{฿}/K\pi_0^\$\right) + rT \pm \frac{1}{2}\lambda^2 T}{\lambda\sqrt{T}} \tag{102}$$

and

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \mathrm{e}^{-\frac{1}{2}z^2}\,\mathrm{d}z\,. \tag{103}$$

Hence, the cryptocurrency exchange-rate option prices can easily be computed numerically. Although for simplicity here we have taken the dollar term structure to be exponential with a constant short rate, it is straightforward to extend the model to allow for calibration to the initial dollar term structure, while preserving the overall tractability of the results. In this respect, our analysis of options on crypto exchange rates can be contrasted with the pioneering work of Madan, Reyners, and Schoutens [27], where the dollar and bitcoin interest rates are taken to be constant (in fact, zero), an assumption that is probably justifiable for the relatively short dated options currently available on the BTC-USD rate, though not very satisfactory, needless to say, from a broader perspective. With the development of interest rate models applicable to bitcoin and other cryptocurrencies, financial institutions will be in a position to trade in cryptocurrency interest rate products. It could be that distributed ledger technologies will eventually find a way of rewarding the holders of positions in virtual currencies with interest on a continuous basis. In the meantime, there should be a role for models of no interest.

Acknowledgments. The authors wish to express their gratitude to J. R. Boland, M. Kecman, and A. Rafailidis for helpful comments.

REFERENCES

[1] S. Nakamoto (2008), *Bitcoin: A Peer-to-Peer Electronic Cash System*, Bitcoin white paper, https://bitcoin.org/bitcoin.pdf.

[2] Bank of International Settlements (2018), *BIS Statistical Bulletin, December* 2018, Monetary and Economic Department, www.bis.org/statistics/bulletin1812.htm.

[3] L. P. Hughston, ed. (1996), *Vasicek and Beyond*, Risk Publications, London.

[4] L. P. Hughston, ed. (2000), *The New Interest Rate Models*, Risk Publications, London.

[5] D. Heath, R. Jarrow, and A. Morton (1992), *Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation*, Econometrica, 60, pp. 77–105.

[6] G. M. Constantinides (1992), *A theory of the nominal term structure of interest rates*, Rev. Financial Stud., 5, pp. 531–552.

[7] D. C. Brody and L. P. Hughston (2004), *Chaos and coherence: A new framework for interest rate modelling*, Proc. Roy. Soc. Lond. A, 460, pp. 85–110.

[8] L. P. Hughston and A. Rafailidis (2005), *A chaotic approach to interest rate modelling*, Finance Stoch., 9, pp. 43–65.

[9] D. C. Brody and L. P. Hughston (2018), *Social discounting and the long rate of interest*, Math. Finance, 28, pp. 306–334.

[10] P. J. Hunt and J. E. Kennedy (2004), *Financial Derivatives in Theory and Practice*, John Wiley & Sons, Chichester.

[11] L. C. G. Rogers (1997), *The potential approach to the term structure of interest rates and foreign exchange rates*, Math. Finance, 7, pp. 157–176.

[12] A. Jobert and L. C. G. Rogers (2002), *Valuations and dynamic convex risk measures*, Math. Finance, 18, pp. 1–22.

[13] G. Johnson and L. L. Helms (1963), *Class D supermartingales*, Bull. Amer. Math. Soc., 69, pp. 59–62.

[14] D. Revuz and M. Yor (1999), *Continuous Martingales and Brownian Motion*, 3rd ed., Springer, Berlin.

[15] D. B. Madan and M. Yor (2006), *Ito's Integrated Formula for Strict Local Martingales*, M. Émery and M. Yor, eds., Séminaire de probabilités XXXIX, Springer, Berlin.

[16] S. Pal and P. Protter (2010), *Analysis of continuous strict local martingales via h-transforms*, Stochastic Process. Appl., 120, pp. 1424–1443.

[17] H. Föllmer and P. Protter (2011), *Local martingales and filtration shrinkage*, ESAIM Probab. Stat., 15, pp. S25–S38.

[18] B. Flesaker and L. P. Hughston (1997), *International models for interest rates and foreign exchange*, Net Exposure, 3, pp. 55–79.

[19] S. Blei and H. J. Engelbert (2009), *On exponential local martingales associated with strong Markov continuous local martingales*, Stochastic Process. Appl., 119, pp. 2859–2880.

[20] A. Mijatović and M. Urusov (2012), *The martingale property of certain local martingales*, Probab. Theory Related Fields, 152, pp. 1–30.

[21] E. Platen and D. Heath (2009), *A Benchmark Approach to Quantitative Finance*, Springer, Berlin.

[22] E. T. Newman, E. Couch, K. Chinnapared, A. Exton, A. Prakash, and R. J. Torrence (1965), *Metric of a rotating charged mass*, J. Math. Phys., 6, pp. 918–919.

[23] J. L. Synge (1972), *A special class of solutions of the Schrödinger equation for a free particle*, Found. Phys., 2, pp. 35–40.

[24] R. Jarrow and P. Protter (2009), *Forward and futures prices with bubbles*, Int. J. Theor. Appl. Finance, 12, pp. 901–924.

[25] R. Jarrow (2016), *Bubbles and multiple-factor asset pricing models*, Int. J. Theor. Appl. Finance, 19, 1650007.

[26] A. Lipton (2001), *Mathematical Methods for Foreign Exchange*, World Scientific, Singapore.

[27] D. B. Madan, S. Reyners, and W. Schoutens (2019), *Advanced model calibration on bitcoin options*, Digital Finance, 1, pp. 117–137.