Determination of the Lévy Exponent
in Asset Pricing Models

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We consider the problem of determining the Lévy exponent in a Lévy model for asset prices given the price data of derivatives. The model, formulated under the real-world measure $\mathbb{P}$, consists of a pricing kernel $\{\pi_t\}_{t\geq 0}$ together with one or more non-dividend-paying risky assets driven by the same Lévy process. If $\{S_t\}_{t\geq 0}$ denotes the price process of such an asset then $\{\pi_t S_t\}_{t\geq 0}$ is a $\mathbb{P}$-martingale. The Lévy process $\{\xi_t\}_{t\geq 0}$ is assumed to have exponential moments, implying the existence of a Lévy exponent $\psi(\alpha) = t^{-1} \log \mathbb{E}(e^{\alpha \xi_t})$ for α in an interval $A \subset \mathbb{R}$ containing the origin as a proper subset. We show that if the prices of power-payoff derivatives, for which the payoff is $H_T = (\zeta_T)^q$ for some time $T > 0$, are given at time 0 for a range of values of q, where $\{\zeta_t\}_{t\geq 0}$ is the so-called benchmark portfolio defined by $\zeta_t = 1/\pi_t$, then the Lévy exponent is determined up to an irrelevant linear term. In such a setting, derivative prices embody complete information about price jumps: in particular, the spectrum of the price jumps can be worked out from current market prices of derivatives. More generally, if $H_T = (S_T)^q$ for a general non-dividend-paying risky asset driven by a Lévy process, and if we know that the pricing kernel is driven by the same Lévy process, up to a factor of proportionality, then from the current prices of power-payoff derivatives we can infer the structure of the Lévy exponent up to a transformation $\psi(\alpha) \to \psi(\alpha + \mu) - \psi(\mu) + c\alpha$, where c and μ are constants.

Keywords: Asset pricing; Lévy models; Lévy processes; Lévy exponent; exponential moments; option pricing; option replication; power payoffs.

I. INTRODUCTION

We are concerned with determining the extent to which derivative prices, taken at time zero, can be used to infer the nature of the underlying jump processes in models where asset prices can move discontinuously. To this end we consider the case of geometric Lévy models, and address the question of to what degree the present values of derivatives can be used to determine the Lévy processes driving the prices of the underlying financial assets. Since the work of Breeden & Litzenberger (1978) and Dupire (1994), a burgeoning literature has developed based on the idea that given the prices of options and other derivatives one can infer distributional and dynamical properties of the price processes of the underlyings. Most of this work deals with continuous price processes. In the present work we consider discontinuous processes and show that in the case of exponential Lévy models the Lévy exponent can be completely determined modulo a two-parameter family of transformations. The paper is structured as follows. In Section II we summarize a few facts concerning Lévy processes. Then we introduce the condition that the Lévy process should admit exponential moments and explore some of the implications of this assumption. In Section III we introduce the

class of geometric Lévy models. These models generalize the standard geometric Brownian motion model. They enable one to see the form that the excess rate of return takes as a function of the level of risk, measured by a volatility parameter σ, and the level of market risk aversion, measured by a parameter λ. In Section IV we state the framework we use for pricing derivatives and indicate how dividends are handled. In Section V we present a method for determining the underlying jump process given a family of power-payoff derivative prices, and the result is summarized in Proposition 1. Then in Section VI we show in Proposition 2 that if the asset on which the derivatives are based is known to be the natural numeraire, then the underlying jump process can be determined with greater precision. We elaborate further on the representation of the Lévy exponent in terms of the prices of power-payoff derivatives and comment in particular on the feasibility of using call option prices for a similar purpose. Finally, in Section VII we establish analogous results for imaginary power-payoff derivatives, where we make use of the techniques of Fourier analysis to show that a general European-style derivative can be expressed as a portfolio of imaginary power payoff derivatives, providing the payoff is smooth and has good asymptotic properties.

II. EXPONENTIAL MOMENTS

We shall assume that the reader is familiar with Lévy processes and their financial applications (Andersen & Lipton 2013, Appelbaum 2004, Bertoin 2004, Brody, Hughston & Mackie 2012, Chan 1999, Cont & Tankov 2004, Gerber & Shiu 1994, Hubalek & Sgarra 2006, Kyprianou 2006, Norberg 2004, Protter 2005, Sato 1999, Schoutens 2004). We mainly work with one-dimensional Lévy processes. For convenience we recall some definitions and classical results. A random process $\{\xi_t\}$ taking values in $\mathbb{R}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a Lévy process if (a) $\xi_{s+t} - \xi_s$ and $\{\xi_u\}_{0 \leq u \leq s}$ are independent for $s, t \geq 0$ (independent increments), (b) $\xi_{s+t} - \xi_s$ has the same law as ξ_t for $s, t \geq 0$ (stationary increments), (c) $\lim_{t \to 0} \mathbb{P}(|\xi_{s+t} - \xi_s| > \epsilon) = 0$ for $\epsilon > 0$ (continuity in probability), and (d) there exists an $\Omega' \in \mathcal{F}$ satisfying $\mathbb{P}(\Omega') = 1$ such that for $\omega \in \Omega'$ the path $\{\xi_t(\omega)\}_{t \geq 0}$ is right-continuous for $t \geq 0$ and has left limits for $t > 0$ (càdlàg property). Note that (b) implies that $\xi_0 = 0$ almost surely. It follows from this definition that for $t \geq 0$ and $\kappa \in \mathbb{R}$ the Fourier transform of ξ_t can be represented in the form

$$\frac{1}{t} \log \mathbb{E}\left[\exp(\mathrm{i}\kappa \xi_t)\right] = \mathrm{i}p\kappa - \frac{1}{2}q\kappa^2 + \int_{-\infty}^{\infty} (\mathrm{e}^{\mathrm{i}\kappa x} - 1 - \mathrm{i}\kappa x \mathbb{1}\{|x| < 1\}) \, \nu(\mathrm{d}x). \tag{1}$$

Here p and $q > 0$ are constants and $\nu(\mathrm{d}x)$ is a Lévy measure. A Borel measure $\nu(\mathrm{d}x)$ on $\mathbb{R}$ is called a Lévy measure if $\nu(\{0\}) = 0$ and

$$\int_{-\infty}^{\infty} 1 \wedge x^2 \, \nu(\mathrm{d}x) < \infty, \tag{2}$$

where $a \wedge b = \min(a, b)$. The Lévy measure associated with a Lévy process has the property that for any measurable set $B \subset \mathbb{R}$ the expected rate at which jumps occur for which the jump size lies in B is $\nu(B)$. The sample paths of a Lévy process have bounded variation on every compact interval of time almost surely if and only if $q = 0$ and

$$\int_{-\infty}^{\infty} 1 \wedge |x| \, \nu(\mathrm{d}x) < \infty. \tag{3}$$

In that case we say that $\{\xi_t\}$ has bounded variation. Let us write $\xi_{t-} = \lim_{s\to t}\xi_s$ for the left limit of the process at time t. The discontinuity at time t is then defined by $\Delta\xi_t = \xi_t - \xi_{t-}$, and for the Lévy measure we have

$$\nu(B) = \frac{1}{t}\,\mathbb{E}\sum_{0\le s\le t}\mathbb{1}\{\Delta\xi_s \in B\} \tag{4}$$

for any $t > 0$. If $\nu(\mathbb{R}) < \infty$ we say that $\{\xi_t\}$ has finite activity, whereas if $\nu(\mathbb{R}) = \infty$ we say that $\{\xi_t\}$ has infinite activity. A necessary and sufficient condition for (3) to hold is that

$$\sum_{0\le s\le t}|\Delta\xi_s| < \infty \tag{5}$$

almost surely for every $t > 0$. If $\sup_t |\Delta\xi_t| \le c$ almost surely for some constant $c > 0$, then we say that $\{\xi_t\}$ has bounded jumps.

In order for $\{\xi_t\}$ to give rise to a Lévy model for asset prices, we require additionally that for every $t > 0$ the random variable ξ_t should satisfy a moment condition of the form

$$\mathbb{E}(\mathrm{e}^{\alpha\xi_t}) < \infty \tag{6}$$

for α in an interval $A = (\beta, \gamma) \subset \mathbb{R}$ containing the origin. Here we set $\beta = \inf \alpha : \mathbb{E}(\mathrm{e}^{\alpha\xi_t}) < \infty$ and $\gamma = \sup \alpha : \mathbb{E}(\mathrm{e}^{\alpha\xi_t}) < \infty$. If a Lévy process satisfies this condition, we say it possesses exponential moments. In that case, it follows (Sato 1999, theorem 25.17) that there exists a so-called Lévy exponent $\psi : \mathbb{C}_A \to \mathbb{R}$, for $\mathbb{C}_A = \{\alpha \in \mathbb{C} \,|\, \mathrm{Re}(\alpha) \in A\}$, such that

$$\mathbb{E}(\mathrm{e}^{\alpha\xi_t}) = \mathrm{e}^{\psi(\alpha)t}, \tag{7}$$

where $\psi(\alpha)$ admits a Lévy-Khinchin representation of the form

$$\psi(\alpha) = p\alpha + \frac{1}{2}q\alpha^2 + \int_{-\infty}^{\infty}(\mathrm{e}^{\alpha x} - 1 - \alpha x\mathbb{1}\{|x| < 1\})\,\nu(\mathrm{d}x). \tag{8}$$

A necessary and sufficient condition for a Lévy process to satisfy (6) for $\alpha \in A$ is that the associated Lévy measure should satisfy

$$\int_{-\infty}^{\infty}\mathrm{e}^{\alpha x}\,\mathbb{1}\{|x| > 1\}\,\nu(\mathrm{d}x) < \infty \tag{9}$$

for $\alpha \in A$ (Sato 1999, theorem 25.3). If $\{\xi_t\}$ admits exponential moments, then one can check that for $p > 0$ we have

$$\mathbb{E}(\,|\xi_t|^p) < \infty. \tag{10}$$

The argument is as follows. Now, for any $\alpha \in \mathbb{R}$ we have

$$\cosh(\alpha\xi_t) = \sum_{k=0}^{\infty}\frac{(\alpha\xi_t)^{2k}}{(2k)!}. \tag{11}$$

Therefore for any $k \in \mathbb{N}$ we have

$$\cosh(\alpha\xi_t) > \frac{(\alpha\xi_t)^{2k}}{(2k)!}. \tag{12}$$

If we choose α so that $|\alpha| < \min(|\beta|, \gamma)$, which ensures that α and $-\alpha$ are in A, then $\mathbb{E}(\cosh(\alpha \xi_t)) < \infty$. Therefore, $\mathbb{E}(|\xi_t|^n) < \infty$ for even $n \in \mathbb{N}$, which implies that $\mathbb{E}(|\xi_t|^p) < \infty$ for all $p \in \mathbb{R}^+$, since for each n and any random variable X it holds that $\mathbb{E}(|X|^n) < \infty$ implies $\mathbb{E}(|X|^p) < \infty$ for $0 \le p \le n$. More generally, for any $\alpha \in A$ and any $p \in \mathbb{R}^+$ we have

$$\mathbb{E}(\mathrm{e}^{\alpha \xi_t} |\xi_t|^p) < \infty. \tag{13}$$

This can be seen as follows. Since A is open, for any $\alpha \in A$ we can choose $\epsilon > 0$ so that $\alpha(1 + \epsilon)$ is still in A. Then by Hölder's inequality we have

$$\mathbb{E}(\mathrm{e}^{\alpha \xi_t} |\xi_t|^p) \le \left(\mathbb{E}(\mathrm{e}^{\alpha(1+\epsilon)\xi_t})\right)^{1/(1+\epsilon)} \left(\mathbb{E}(|\xi_t|^{p(1+\epsilon)/\epsilon})\right)^{\epsilon/(1+\epsilon)}. \tag{14}$$

But we have already established that the terms on the right are finite, and that gives (13). A similar argument shows that if $\{\xi_t\}$ admits exponential moments then

$$\int_{-\infty}^{\infty} \mathrm{e}^{\alpha x} |x|^p \, \mathbb{1}\{|x| > 1\} \, \nu(\mathrm{d}x) < \infty, \tag{15}$$

for $\alpha \in A$ and $p > 0$. Setting $\alpha = 0$ and $p = 1$, we see in particular that

$$\int_{-\infty}^{\infty} |x| \, \mathbb{1}\{|x| > 1\} \, \nu(\mathrm{d}x) < \infty, \tag{16}$$

which implies that one can extend the truncated term on the right side of (8) to an integral of the form $\int |x| \, \nu(\mathrm{d}x)$, over the whole of $\mathbb{R}$, by dropping the indicator function and redefining the constant p in equation (8). The finiteness of integrals (13) and (15) allows one to compute the Greeks for various derivative payouts in exponential Lévy models.

Examples of Lévy processes admitting exponential moments include: (a) Brownian motion, for which $\psi(\alpha) = \frac{1}{2}\alpha^2$, $\alpha \in \mathbb{R}$; (b) the Poisson process with rate m, for which $\psi(\alpha) = m(\mathrm{e}^{\alpha} - 1)$ and $\nu(\mathrm{d}z) = m\delta_1(\mathrm{d}z)$; (c) the compound Poisson process with rate m, for which $\psi(\alpha) = m(\theta(\alpha) - 1)$, where $\theta(\alpha)$ is the moment generating function for the distribution $\mu(\mathrm{d}x)$ of a typical jump and $\nu(\mathrm{d}z) = m\mu(\mathrm{d}x)$; (d) the gamma process with rate m, for which $\psi(\alpha) = -m\log(1 - \alpha)$, $\alpha < 1$, where $\nu(\mathrm{d}z) = \mathbb{1}\{z > 0\} \, z^{-1} \exp(-z) \, \mathrm{d}z$ (Dickson & Waters 1993, Heston 1993, Brody, Macrina & Hughston 2008, Yor 2007); (e) the variance gamma (VG) process, for which $\psi(\alpha) = -m\log(1 - \alpha^2/2m^2)$, where we have $-2^{1/2}m < \alpha < 2^{1/2}m$ (Madan & Seneta 1990, Madan & Milne 1991, Madan, Carr & Chang 1998); (f) the truncated stable family of Lévy processes, which includes the gamma process and the VG process as special cases (Koponen 1995, Carr, Geman, Madan & Yor 2002, Andersen & Lipton 2013, Küchler & Tappe 2014); (g) hyperbolic processes (Eberlein & Keller 1995, Eberlein, Keller & Prause 1998, Bingham & Kiesel 2001); (h) generalized hyperbolic processes (Eberlein 2001); (i) normal inverse Gaussian processes (Barndorff-Nielsen 1998); and (j) Meixner processes (Schoutens & Teugels 1998).

III. GEOMETRIC LÉVY MODEL

The geometric Lévy model for asset prices can be viewed as an extension of the well-known geometric Brownian motion model to the Lévy regime. For simplicity, we consider a model driven by a one-dimensional process $\{\xi_t\}_{t \ge 0}$. The generalization to higher dimensional Lévy

processes is straightforward. We assume that $\{\xi_t\}$ admits exponential moments and denote the associated Lévy exponent by $\{\psi(\alpha)\}_{\alpha \in A}$ for $A = (\beta, \gamma)$ with $\beta < 0 < \gamma$. The process $\{m_t\}_{t \geq 0}$ defined by

$$m_t = \mathrm{e}^{\alpha \xi_t - \psi(\alpha)t}, \tag{17}$$

for some choice of $\alpha \in A$, is the corresponding geometric Lévy martingale with volatility α. By the stationary and independent increments properties we find that $\mathbb{E}_s(m_t) = m_s$. Here we write $\mathbb{E}_t(\cdot) = \mathbb{E}(\cdot \,|\, \mathcal{F}_t)$, where $\{\mathcal{F}_t\}$ denotes the augmented filtration generated by $\{\xi_t\}$. The associated geometric Lévy model consists of a pricing kernel, a money market account, and one or more so-called investment-grade assets. See Duffie (1992), Hunt & Kennedy (2004), Cochrane (2005) for general aspects of the theory of pricing kernels in arbitrage-free asset pricing models. For the construction of the pricing kernel $\{\pi_t\}_{t \geq 0}$ in the context of a Lévy model we let $r \in \mathbb{R}$ and $\lambda > 0$ be constants, and assume that $-\lambda \in A$. Then we set

$$\pi_t = \mathrm{e}^{-rt - \lambda \xi_t - \psi(-\lambda)t}. \tag{18}$$

We refer to the related process $\{\zeta_t\}_{t \geq 0}$ defined by $\zeta_t = 1/\pi_t$ as the growth-optimal portfolio or *natural numeraire* asset (Flesaker & Hughston 1997). It serves as a benchmark relative to which other non-dividend-paying assets are martingales. In some calculations it is convenient to make reference to the natural numeraire instead of the pricing kernel. The money market account $\{B_t\}_{t \geq 0}$ is taken to have the value $B_t = B_0 \mathrm{e}^{rt}$ at time t, where B_0 denotes its initial value in some fixed unit of account. The idea of an investment-grade asset is that it should offer a rate of return that is strictly greater than the interest rate. Ordinary stocks and bonds are in this sense investment-grade, whereas put options and short positions in ordinary stocks and bonds are not. We assume for the moment that the assets pay no dividends over the time horizons considered (dividends will be treated shortly), and we write $\{S_t\}_{t \geq 0}$ for the value process of a typical non-dividend paying risky asset in the geometric Lévy model. We require that the product of the pricing kernel and the asset price should be a martingale, which we take to be geometric Lévy martingale of the form

$$\pi_t S_t = S_0 \mathrm{e}^{\beta \xi_t - \psi(\beta)t} \tag{19}$$

for some $\beta \in A$. From the formulae above we deduce that

$$S_t = S_0 \, \mathrm{e}^{rt + \sigma \xi_t + \psi(-\lambda)t - \psi(\sigma - \lambda)t}, \tag{20}$$

where $\sigma = \beta + \lambda$. We assume that $\sigma > 0$ and that $\sigma \in A$. It follows that the price can be expressed in the form

$$S_t = S_0 \, \mathrm{e}^{rt + R(\lambda, \sigma)t + \sigma \xi_t - \psi(\sigma)t}, \tag{21}$$

where

$$R(\lambda, \sigma) = \psi(\sigma) + \psi(-\lambda) - \psi(\sigma - \lambda). \tag{22}$$

Thus we see that σ is the volatility of the asset price relative to the given Lévy process and that $R(\lambda, \sigma)$ is the excess rate of return above the interest rate. The parameter λ can be interpreted as a measure of the level of market risk aversion. A calculation shows that the excess rate of return is bilinear in λ and σ if and only if $\{\xi_t\}$ is a Brownian motion

(Brody *et al* 2012). It follows that the interpretation of λ as a "market price of risk", which is valid for models based on a Brownian filtration, does not carry through directly to the general Lévy regime. Nevertheless, the notion of excess rate of return is well defined, and under the assumptions that we have made the strict convexity of the Lévy exponent implies that the excess rate of return is strictly positive. To show that $R(\lambda, \sigma) > 0$ we can use the Lévy-Khinchin formula (8) to deduce that

$$R(\lambda, \sigma) = \int_{-\infty}^{\infty} (\mathrm{e}^{\sigma x} - 1)(1 - \mathrm{e}^{-\lambda x})\, \nu(\mathrm{d}x). \tag{23}$$

It follows by inspection of (23) that the excess rate of return is an increasing function of the volatility and the level of risk aversion.

In the case of a single asset driven by a single Lévy process one can without loss of generality set $\sigma = 1$. This can be achieved by defining a rescaled Lévy process $\{\bar{\xi}_t\}$ by setting $\bar{\xi}_t = \sigma \xi_t$. Then we define $\bar{\psi}(\alpha) = \psi(\sigma\alpha)$ and set $\bar{\lambda} = \lambda/\sigma$, and we have

$$\pi_t = \mathrm{e}^{-rt-\bar{\lambda}\bar{\xi}_t-\bar{\psi}(-\bar{\lambda})t}, \quad S_t = S_0\, \mathrm{e}^{rt+\bar{R}(\bar{\lambda},1)t+\bar{\xi}_t-\bar{\psi}(1)t}, \tag{24}$$

where $\bar{R}(\bar{\lambda}, 1) = \bar{\psi}(1) + \psi(-\bar{\lambda}) - \bar{\psi}(1 - \bar{\lambda})$. If we then drop the bars, we are led back to a version of the model already set up, but with $\sigma = 1$. Nevertheless, it can be helpful to leave the parameter σ intact as part of the theory, since there are situations where one would like to compare versions of the model for different values of the parameter, e.g. for sensitivity analysis and calculations of the Greeks. On the other hand, there are also situations where it is desirable to make use of simplifications resulting from setting $\sigma = 1$; an example of this can be found in the proof of Proposition 1. It should be noted that the value of the asset given by (21) does not depend on the drift of the Lévy process, for if we replace ξ_t with $\xi_t + \epsilon t$ for some $\epsilon \in \mathbb{R}$ then the Lévy exponent $\psi(\alpha)$ gets replaced with $\psi(\alpha) + \epsilon\alpha$, and the combination $\sigma\xi_t - \psi(\sigma)t$ appearing in the formula for the asset price is left unchanged. This has the implication that if one attempts to determine the Lévy exponent from the prices of derivatives, one will be left with an indeterminacy of the form $\psi(\alpha) \to \psi(\alpha) + c\alpha$ for some unknown constant c.

It is worth recalling that one of the motivations indicated by Mandelbrot (1963) for the introduction of Lévy models in finance is the possibility of offering an explanation for the apparent existence of "fat tails" in the distributions of returns. But it seems that what he had in mind was not the construction of specific dynamical models for price processes, but rather the introduction of infinitely-divisible distributions with infinite moments to model the returns on such assets, an assumption that makes the construction of dynamical models difficult. From an empirical point of view, however, the requirement a Lévy process should have "thin tails" is a relatively mild one: a sufficient condition for a Lévy process to admit exponential moments (and hence to have thin tails) is that the jumps should be bounded (Protter 2005, theorem 34), even if the bounds are set at arbitrarily high values. Thus, from a modern point of view the use of Lévy processes in finance stems not so much from a desire to model the distributions of the tails of returns but rather to account for the characteristics of the jumps that asset prices can undertake.

IV.   DERIVATIVE PRICING

The price H_0 at time 0 of a European style derivative that delivers a single random payment H_T at time T is given by

$$H_0 = \mathbb{E}(\pi_T H_T). \qquad (25)$$

The expectation is, of course, with respect to the real-world probability measure. The pricing kernel takes care of both the discounting and the probability weighting needed to give the answer. The use of such a formula for derivative pricing is well known, but it may be useful to recall the argument. In the general theory of asset pricing one fixes, as we have done, a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is interpreted as the real-world measure, together with a filtration $\{\mathcal{F}_t\}$, and one assumes the existence of a pricing kernel $\{\pi_t\}_{t \geq 0}$ satisfying $\pi_t > 0$ for all $t \geq 0$ such that for any asset with a non-negative price process $\{S_t\}_{t \geq 0}$ and a non-decreasing cumulative dividend process $\{K_t\}_{t \geq 0}$, the associated deflated gain process $\{\bar{S}_t\}_{t \geq 0}$ defined by

$$\bar{S}_t = \pi_t S_t + \int_0^t \pi_s \, dK_s \qquad (26)$$

is a martingale. So far, we have considered limited liability assets, for which the prices are non-negative and the cumulative dividend process is increasing. By a general asset, not necessarily of limited liability, we mean an asset with the property that its price can be expressed as the difference between the prices of two limited liability assets, and for which the cumulative dividend process can be expressed as the difference between two increasing cumulative dividend processes. It follows then that the deflated gain process of a general asset is also a martingale.

The formula given above allows for the possibility of both continuously paid and discretely paid dividends. If the dividends are entirely discrete (and paid at possibly random times), then the deflated gain process can be expressed in the form

$$\bar{S}_t = \pi_t S_t + \sum_{0 \leq s \leq t} \pi_s \, \Delta(K_s). \qquad (27)$$

At each time at which a dividend is paid, the cumulative dividend process jumps, and the value of the jump is equal to the dividend. In the case of a European style derivative with a single payoff H_T made at time T, we think of the payoff as a dividend, and hence the cumulative dividend process is zero up to time T, then jumps to H_T at T. The sum in (27) reduces to a single term, given by the jump $\Delta(K_T) = H_T$, and we have $\bar{S}_T = \pi_T H_T$ at T. Since the value of the derivative itself drops to zero the instant that the dividend is paid, it follows by the martingale condition that $\bar{S}_0 = \mathbb{E}(\pi_T H_T)$, which gives us (25). In the literature, by virtue of a conventional abuse of notation, one often writes the price process of the derivative in the form $\{H_s\}_{0 \leq s \leq T}$, as if somehow the terminal value of the derivative is what is paid; in reality, the value of the derivative itself at maturity is 0, whereas the payoff (or dividend) at time T is H_T. This is consistent with the notion that the price process of the derivative should be right continuous with left limits.

We are also in a position to check that if a risky asset pays a proportional dividend at the constant rate δ then its price in the geometric Lévy model will be given by

$$S_t = S_0 \, e^{(r-\delta)t + R(\lambda,\sigma)t + \sigma \xi_t - \psi(\sigma)t}. \qquad (28)$$

The expression is of course intuitively plausible, perhaps even obvious, by analogy with the corresponding result in the geometric Brownian motion model. Nevertheless, we need to check that the process $\{\bar{S}_t\}$ defined by (26) is a martingale. A calculation making use of equations (18), (26) and (28) gives

$$\bar{S}_t = S_0 \, \mathrm{e}^{-\delta t + (\sigma - \lambda)\xi_t - \psi(\sigma - \lambda)t} + \delta S_0 \int_0^t \mathrm{e}^{-\delta u + (\sigma - \lambda)\xi_u - \psi(\sigma - \lambda)u} \mathrm{d}u. \tag{29}$$

Splitting the integral at time $s < t$ and taking a conditional expectation making use of Fubini's theorem, we get

$$\mathbb{E}_s \bar{S}_t = S_0 \, \mathrm{e}^{-\delta t + (\sigma - \lambda)\xi_s - \psi(\sigma - \lambda)s} + \delta S_0 \int_0^s \mathrm{e}^{-\delta u + (\sigma - \lambda)\xi_u - \psi(\sigma - \lambda)u} \mathrm{d}u$$

$$+ \delta S_0 \int_s^t \mathrm{e}^{-\delta u + (\sigma - \lambda)\xi_s - \psi(\sigma - \lambda)s} \mathrm{d}u, \tag{30}$$

from which it follows immediately that $\mathbb{E}_s \bar{S}_t = \bar{S}_s$. Thus we have shown that the price defined by (28) together with the proportional dividend rate δ is such that the resulting deflated gain process is a martingale, which demonstrates that (28) is indeed the correct expression in the geometric Lévy model for the price of a risky asset that pays a proportional dividend at a constant rate.

V.   DETERMINATION OF THE LÉVY EXPONENT

Turning to the problem of the determination of the Lévy exponent from price data, we proceed to consider a one-parameter family of so-called power-payoff derivatives, for which

$$H_T = (S_T)^q, \tag{31}$$

where $q \in \mathbb{R}$. Various authors have considered the pricing of similar structures in Lévy models relating the Lévy exponent to the price (see, e.g., Carr & Lee 2009, Fajardo 2018, and references cited therein). The value $H_0 \in \mathbb{R}^+ \cup \infty$ of a power-payoff derivative at time zero is given by

$$H_0 = \mathbb{E}(\pi_T S_T^q). \tag{32}$$

Here we allow the possibility that the value of the derivative may not be finite for some values of q. In a model driven by Brownian motion, the asset price takes the form

$$S_T = S_0 \, \mathrm{e}^{(r + \lambda \sigma)T + \sigma W_T - \frac{1}{2}\sigma^2 T}, \tag{33}$$

and for the pricing kernel we have

$$\pi_T = \mathrm{e}^{-rT - \lambda W_T - \frac{1}{2}\lambda^2 T}, \tag{34}$$

where for convenience we set $\pi_0 = 1$. It follows that

$$\pi_T S_T^q = S_0^q \, \mathrm{e}^{(q-1)rT + q\lambda\sigma T + (q\sigma - \lambda)W_T - \frac{1}{2}(q\sigma^2 + \lambda^2)T}. \tag{35}$$

A calculation then allows one to deduce that the value of the power payoff derivative, regarded as a function q, takes the form

$$H_0(q) = S_0^q \, e^{(q-1)rT + \frac{1}{2}q(q-1)\sigma^2 T} \tag{36}$$

in the Brownian case. One notes that the terms involving λ cancel when the expectation is taken, so the value of the derivative only depends on S_0, r, σ, and q, and that $H_0(q)$ is finite for all values of q.

In the case of a geometric Lévy model, the asset price is given by (21) and the pricing kernel is given by (18). Thus we have

$$\pi_T S_T^q = S_0^q \, e^{(q-1)rT + (q\sigma - \lambda)\xi_T + (q-1)\psi(-\lambda)T - q\psi(\sigma - \lambda)T}. \tag{37}$$

The value of the power-payoff derivative regarded as a function of q then takes the form

$$H_0(q) = S_0^q \, e^{(q-1)rT + \psi(q\sigma - \lambda)T + (q-1)\psi(-\lambda)T - q\psi(\sigma - \lambda)T}. \tag{38}$$

It is perhaps remarkable that an explicit expression is obtained, but this allows one to study in detail the relation between the type of Lévy model under consideration and the resulting values of derivatives. We note that $H_0(0) = e^{-rT}$ and that $H_0(1) = S_0$, as one would expect, and it should be evident that in general $H_0(q)$ is finite only for a certain range of values of the parameter q. In particular, suppose that $\sigma > \lambda > 0$ and that $\sigma \in A$ and $-\lambda \in A$. Then for $A = (\beta, \gamma)$ clearly $q\sigma - \lambda \in A$ if and only if

$$\frac{1}{\sigma}(\beta + \lambda) < q < \frac{1}{\sigma}(\gamma + \lambda). \tag{39}$$

Since $\beta < -\lambda$ and $\gamma > 0$, these inequalities ensure that the interior of the set of values of q for which $H_0(q) < \infty$ is an open set B that includes the origin.

Now we are in a position to ask to what extent specification of the family of derivative prices $\{H_0(q)\}_{q \in B}$ allows one to infer the nature of the Lévy process driving the model. To this end we note that from observations of $H_0(0)$ and $H_0(1)$ one can infer the value of r and S_0. Thus without loss of generality it suffices to regard the function

$$D_0(q) = \frac{1}{T} \log \frac{H_0(q)}{S_0^q \, e^{(q-1)rT}} = \psi(q\sigma - \lambda) + (q-1)\psi(-\lambda) - q\psi(\sigma - \lambda), \tag{40}$$

which is finite for $q \in B$, as representing the data supplied by the family of derivative prices.

Proposition 1 *Let the prices of derivatives with payoff payoffs $H_T = (S_T)^q$ for $q \in \mathbb{R}$ be given for a non-dividend-paying risky asset $\{S_t\}_{t \geq 0}$ that is known to be a geometric Lévy process, and suppose it is known that the pricing kernel is a geometric Lévy process driven by the same Lévy process up to a factor of proportionality. Then the Lévy exponent can be inferred up to a transformation $\psi(\alpha) \to \psi(\alpha + \mu) - \psi(\mu) + c\alpha$, where c and μ are constants.*

Proof. Without loss of generality one can set $\sigma = 1$. Then we have

$$D_0(q) = \psi(q - \lambda) + (q-1)\psi(-\lambda) - q\psi(1 - \lambda). \tag{41}$$

In the setting of the problem we take $D_0(q)$ to be given for all $q \in \mathbb{R}$ and finite in some open set B, and we consider λ to be unknown. The goal is to determine the Lévy exponent.

Writing $\tilde{\psi}(\alpha) = \psi(\alpha - \lambda) - \psi(-\lambda)$, we have $D_0(q) = \tilde{\psi}(q) - q\tilde{\psi}(1)$. This implies that $\tilde{\psi}(q) = D_0(q) + qb$ for some $b \in \mathbb{R}$. Now, it is easy to see that $\psi(\alpha) = \tilde{\psi}(\alpha + \lambda) - \tilde{\psi}(\lambda)$. We conclude that for some choice of λ and b the Lévy exponent takes the form

$$\psi(\alpha) = D_0(\alpha + \lambda) - D_0(\lambda) + b\alpha. \tag{42}$$

Substituting (42) into the right-hand side of (41), one can check that the solution is valid. Finally, we note that under a transformation of the form $\psi(\alpha) \to \hat{\psi}(\alpha)$, with $\hat{\psi}(\alpha) = \psi(\alpha + \mu) - \psi(\mu) + c\alpha$, where $c, \mu \in \mathbb{R}$, we find that $\hat{\psi}(\alpha) = D_0(\alpha + \lambda + \mu) - D_0(\lambda + \mu) + b\alpha$. The effect of the transformation is $\lambda \to \hat{\lambda} = \lambda + \mu$. Since λ is unknown, this shows that the Lévy exponent is determined only up to a transformation of the type indicated. $\square$

VI. INTERPRETIVE REMARKS

Following on from this result, a few comments may be in order. We recall that a Lévy process is completely characterized by the random variable ξ_t at a single instant of time t. This reflects the fact that there is a one-to-one correspondence between Lévy processes and infinitely divisible distributions, and a Lévy process has the property that the distribution of its value at any particular time is infinitely divisible. Taking $t = 1$ for convenience, we have $\psi(\alpha) = \log \mathbb{E}[e^{\alpha \xi_1}]$ in the case of a Lévy process that admits exponential moments, and we note that the distribution of ξ_1 is determined by the Lévy exponent $\{\psi(\alpha)\}_{\alpha \in A}$. Each such distribution belongs in a natural way to a certain one-parameter family of distributions, which we call an Esscher family of distributions. The distribution of ξ_1 is the function $F : \mathbb{R} \to [0, 1]$ defined by $F(x) = \mathbb{E}[\mathbb{1}\{\xi_1 \le x\}]$. For the associated Lévy exponent we have

$$\psi(\alpha) = \log \int_{-\infty}^{+\infty} e^{\alpha x} \, \mathrm{d}F(x). \tag{43}$$

The corresponding Esscher family $F_\delta : \mathbb{R} \to [0, 1]$, $\delta \in A$, is defined by the measure change

$$F_\delta(x) = \mathbb{E}[e^{\delta \xi_1 - \psi(\delta)} \mathbb{1}\{\xi_1 \le x\}]. \tag{44}$$

We may accordingly ask for the structure of the Lévy exponent associated with F_δ. This is

$$\psi_\delta(\alpha) = \log \int_{-\infty}^{+\infty} e^{\alpha x} \, \mathrm{d}F_\delta(x) = \log \int_{-\infty}^{+\infty} e^{\delta x - \psi(\delta)} e^{\alpha x} \, \mathrm{d}F(x) = \psi(\alpha + \delta) - \psi(\delta). \tag{45}$$

So we see that by Proposition 1, the specification of the prices of power-payoff derivatives allows one to determine the Esscher family of the Lévy exponent of the underlying Lévy process, modulo an irrelevant linear term. Lévy processes that are equivalent in this sense can be said to belong to the same "noise type" (Brody, Hughston & Yang 2013).

On the other hand, if more information is known *a priori* about the nature of the underlying asset, then a more precise determination of the Lévy exponent is possible. Consider the case, for instance, where it is known that the asset on which the power payoff derivative is based is the natural numeraire. Then $\sigma = \lambda$, and for the asset price we have

$$\zeta_t = \zeta_0 \, e^{rt + R(\lambda, \lambda)t + \lambda \xi_t - \psi(\lambda)t}, \tag{46}$$

where the excess rate of return is given by $R(\lambda, \lambda) = \psi(\lambda) + \psi(-\lambda)$. In this case we can without loss of generality set $\lambda = 1$. It follows then from (41) and (42) that $D_0(1) = 0$ and hence $\psi(\alpha) = D_0(\alpha + 1) + b\alpha$. Thus we have shown the following:

Proposition 2 *Let the prices of derivatives with power payoffs $H_T = (\zeta_T)^q$ for $q \in \mathbb{R}$ be given for a natural numeraire $\{\zeta_t\}_{t \geq 0}$ that is known to be a geometric Lévy process. Then the Lévy exponent can be inferred up to a transformation of the form $\psi(\alpha) \to \psi(\alpha) + b\alpha$, where b is a constant.*

In a geometric Lévy model, the pricing kernel can be written in the form $\pi_t = e^{-rt}\Lambda_t$ where the martingale $\{\Lambda_t\}_{t \geq 0}$ defined by

$$\Lambda_t = e^{-\lambda \xi_t - \psi(-\lambda)t} \tag{47}$$

determines a change of measure. Thus, for any $\mathcal{F}_t$-measurable random variable Z_t we have

$$\tilde{\mathbb{P}}(Z_t < z) = \tilde{\mathbb{E}}\left[\mathbb{1}(Z_t < z)\right] = \mathbb{E}\left[\Lambda_t \,\mathbb{1}(Z_t < z)\right]. \tag{48}$$

We refer to $\tilde{\mathbb{P}}$ as the risk-neutral measure and write $\tilde{\mathbb{E}}$ for expectation under $\tilde{\mathbb{P}}$. The terminology "risk-neutral" comes from the fact that $\tilde{\mathbb{E}}(S_t) = S_0 e^{rt}$ in the geometric Lévy model. Then $\{\tilde{\psi}(a)\}$ has the interpretation of being the Lévy exponent associated with ξ_t under the risk-neutral measure. That is to say,

$$\tilde{\psi}(a) = \frac{1}{t} \log \tilde{\mathbb{E}}[e^{a\xi_t}]. \tag{49}$$

The essence of Proposition 1 is that the family of derivative prices can be used to calculate $\{\tilde{\psi}(a)\}$, which fixes the exponent $\{\psi(a)\}$ under $\mathbb{P}$, modulo the freedom indicated.

Let us write $C_{0T}(x)$ for the price at time 0 of a call option with maturity T and strike x. Can one use the data $\{C_{0T}(x)\}_{x \geq 0}$ for fixed T to ascertain the Lévy exponent in a geometric Lévy model? The answer is yes, though the method is less straightforward than the use of power-payoff derivatives, as we shall see. Now, it is known that if the random variable S_T corresponding to the terminal value of the asset at time T admits a risk-neutral density function, then we can use the idea of Breeden & Litzenberger (1978) to work out this density in terms of call option data. In particular, if we write

$$\tilde{\theta}(x) = \frac{\mathrm{d}}{\mathrm{d}x} \tilde{\mathbb{P}}(S_T \leq x) \tag{50}$$

for the density of S_T under the risk-neutral measure $\tilde{\mathbb{P}}$, we have

$$\tilde{\theta}(x) = e^{-rT} \frac{\mathrm{d}^2 C_{0T}(x)}{\mathrm{d}x^2}. \tag{51}$$

This follows from the fact that

$$C_{0T}(x) = e^{-rT} \int_0^\infty (y - x)^+ \,\tilde{\theta}(y) \,\mathrm{d}y. \tag{52}$$

Then $\{\tilde{\theta}(x)\}_{x \geq 0}$ can be used to calculate the values of power-payoff derivatives via the relation

$$\tilde{\mathbb{E}}(S_T^q) = \int_0^\infty x^q \,\tilde{\theta}(x) \,\mathrm{d}x, \tag{53}$$

and from there we can work out $\tilde{\psi}(\alpha)$, as indicated in the previous section. The difficulty with this approach is that in a geometric Lévy model the distribution of S_T does not in general admit a density function, and the system of call option prices $\{C_{0T}(x)\}_{x \geq 0}$ is not differentiable for all $x \in \mathbb{R}^+$. The situation can be remedied to some extent if instead we make use of the risk-neutral distribution function $\{\tilde{F}(x)\}_{x \geq 0}$ and express the option price in the form of a Lebesgue-Stieltjes integral, writing

$$C_{0T}(x) = e^{-rT} \int_0^\infty (y - x)^+ \, \mathrm{d}\tilde{F}(y), \tag{54}$$

with the understanding that the distribution function is right-continuous. Then the derivative of the option price with respect to the strike is defined for all $x \in \mathbb{R}^+$ apart from points of discontinuity of the distribution function, and this is sufficient to enable us to recover the distribution function in its entirety. Once the distribution function is known, one can determine the Lévy exponent by calculating the system of power-payoff prices, given for $q \in \mathbb{R}$ by

$$H_0(q) = e^{-rT} \int_0^\infty x^q \, \mathrm{d}\tilde{F}(x). \tag{55}$$

VII.   IMAGINARY POWER PAYOFFS

As another example of a one-parameter family of derivatives from which information can be extracted concerning the Lévy exponent when the underlying is a geometric Lévy asset we consider a family of imaginary power payoffs, for which the terminal cash flow is given by

$$F_T(q) = (S_T)^{iq}, \tag{56}$$

where $q \in \mathbb{R}$. The value of such a contract at time zero takes the form

$$F_0(q) = \mathbb{E}(\pi_T S_T^{iq}) = \mathbb{E}(\pi_T e^{iq \log S_T}). \tag{57}$$

Since the payoff is a complex function, we are in effect valuing two different derivatives, each with its own payoff. That is to say,

$$F_0(q) = \mathbb{E}[\pi_T \cos(q \log S_T)] + i \, \mathbb{E}[\pi_T \sin(q \log S_T)]. \tag{58}$$

Thus $\{F_0(q)\}$, $q \in \mathbb{R}$, can be thought of as a pair of price families $\{F_0^c(q)\}$ and $\{F_0^s(q)\}$, for which the corresponding payoff functions are given respectively by $F_T^c(q) = \cos(q \log S_T)$ and $F_T^s(q) = \sin(q \log S_T)$. Note that the payoffs, and hence the prices, are bounded for all values of q. A calculation then shows that

$$F_0(q) = S_0^{iq} \, e^{r(iq-1)T} \, e^{(-iq \, \psi(\sigma - \lambda) + (iq-1)\psi(-\lambda) + \psi(iq\sigma - \lambda))T}. \tag{59}$$

With these ideas at hand we can use the methods of Fourier analysis to investigate more general payoffs. We begin by recalling briefly a few well known facts. Let the map $f : \mathbb{R} \to \mathbb{R}$ be such that $f \in L^1$. The Fourier transform of f is the function $g : \mathbb{R} \to \mathbb{C}$ defined by

$$g(q) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-iqx} f(x) \, \mathrm{d}x. \tag{60}$$

Under various further conditions the relation between f and g can then be inverted. For example, if $f \in L^1$ and is continuous, and if $g \in L^1$, then

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iqx} g(q) \, \mathrm{d}q. \tag{61}$$

A sufficient condition for these requirements to be satisfied is that f should be a "good" function in the sense of Lighthill (1958), that is to say, that it should be everywhere differentiable any number of times and such that it and all its derivatives are $O\left(|x|^{-n}\right)$ as $|x| \to \infty$ for all $n \in \mathbb{N}$. We recall that $f(x)$ is said to be $O(h(x))$ as $x \to \infty$ if

$$\limsup_{x \to \infty} \left| \frac{f(x)}{h(x)} \right| < \infty. \tag{62}$$

If f is a good function then its Fourier transform g is also good. Now consider the situation where the payoff of a European-style derivative with value H_0 at time zero takes the form $H_T = f(\log S_T)$ for some $f \in L^1$. If f is continuous and $g \in L^1$, then by use of (61) we can write the payoff in the form

$$H_T = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iq \log S_T} g(q) \, \mathrm{d}q. \tag{63}$$

This expresses H_T as a portfolio of imaginary power payoffs parameterized by q, where $g(q)$ determines the relative portfolio weighting for the given q. Multiplying each side of equation (63) with the pricing kernel π_T and taking the expectation we obtain

$$\mathbb{E}(\pi_T H_T) = \frac{1}{\sqrt{2\pi}} \mathbb{E}\left(\int_{-\infty}^{\infty} \pi_T \, e^{iq \log S_T} g(q) \, \mathrm{d}q \right), \tag{64}$$

from which it follows that

$$\mathbb{E}(\pi_T H_T) = \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{\infty} \mathbb{E}\left[\pi_T e^{iq \log S_T} \right] g(q) \, \mathrm{d}q \right). \tag{65}$$

Inserting (57) into (65), we then have

$$H_0 = \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{\infty} F_0(q) \, g(q) \, \mathrm{d}q \right). \tag{66}$$

Thus the price of the derivative can be expressed as the value of a portfolio of imaginary power payoff derivatives. To check that the interchange of the expectation and the integration in equation (64) is valid (Kingman & Taylor 1966, Theorem 6.5), we note that

$$\mathbb{E}\left(\int_{-\infty}^{\infty} \pi_T \, e^{iq \log S_T} g(q) \, \mathrm{d}q \right) = \int_{\omega \in \Omega} \int_{-\infty}^{\infty} \pi_T \, e^{iq \log S_T} g(q) \, \mathrm{d}q \, \mathbb{P}(\mathrm{d}\omega), \tag{67}$$

and that

$$\int_{\omega \in \Omega} \int_{-\infty}^{\infty} \left| \pi_T \, e^{iq \log S_T} g(q) \right| \mathrm{d}q \, \mathbb{P}(\mathrm{d}\omega) = \mathbb{E}[\pi_T] \int_{-\infty}^{\infty} |g(q)| \, \mathrm{d}q < \infty. \tag{68}$$

As an example we consider a European-style derivative payoff $H_T = f(\log S_T)$ at time T for which f takes the form of a normal density function

$$f(x) = \frac{1}{\sqrt{2\pi u}} e^{-\frac{1}{2} \frac{(x-a)^2}{u}}, \tag{69}$$

with mean a and variance u. In the case of a geometric Brownian motion model, the random variable corresponding to the terminal value of the underlying asset is normally distributed with a risk-neutral density of the form

$$\tilde{\theta}(x) = \frac{1}{\sqrt{2\pi v}} e^{-\frac{1}{2} \frac{(x-b)^2}{v}}, \tag{70}$$

where $b = \left(r - \frac{1}{2}\sigma^2\right) T$ and $v = \sigma^2 T$. The price of the derivative at time zero is given by

$$H_0 = e^{-rT} \tilde{\mathbb{E}}\left[H_T\right] = e^{-rT} \int_{-\infty}^{\infty} \tilde{\theta}(x) f(x) \, \mathrm{d}x. \tag{71}$$

With some calculation, we find that

$$H_0 = e^{-rT} \frac{1}{\sqrt{2\pi(u + v)}} e^{-\frac{1}{2} \frac{(a-b)^2}{u+v}}. \tag{72}$$

For instance, if we set $a = 0, u = 1$ and insert the aforementioned values of b and v, we obtain

$$H_0 = e^{-rT} \frac{1}{\sqrt{2\pi(1 + \sigma^2 T)}} e^{-\frac{1}{2} \frac{(r - \frac{1}{2}\sigma^2)^2 T^2}{1 + \sigma^2 T}}. \tag{73}$$

Alternatively, we can replicate the payoff of the derivative as a portfolio of imaginary power payoffs using the Fourier technique. Since $f \in L^1$, we can set

$$g(q) = \frac{1}{2\pi\sqrt{u}} \int_{-\infty}^{\infty} e^{-iqx} e^{-\frac{1}{2} \frac{(x-a)^2}{u}} \, \mathrm{d}x, \tag{74}$$

and a calculation gives

$$g(q) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}q^2 u} e^{-iqa}. \tag{75}$$

By (63) and (75), and using the fact that f is a good function, one sees that the payoff of the derivative can be expressed in the form

$$H_T = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_T^{iq} e^{-\frac{1}{2}q^2 u} e^{-iqa} \mathrm{d}q. \tag{76}$$

Then by (66) we obtain the derivative price as a portfolio of imaginary power-payoff prices:

$$H_0 = \frac{1}{2\pi} \left(\int_{-\infty}^{\infty} F_0(q) \, e^{-\frac{1}{2}q^2 u} e^{-iqa} \, \mathrm{d}q \right). \tag{77}$$

We observe, finally, that if the prices of imaginary power payoff derivatives delivering the cash flows defined by (56) are given for all $q \in \mathbb{R}$, then by adapting the framework of Proposition 1 we can work out the implied Lévy exponent modulo some specified freedom. We recall that the price of a power-payoff derivative at time zero is given for power q by equation (59). Therefore if we consider the function $\{D_0(q)\}_{q \in \mathbb{R}}$ defined by

$$D_0(q) = \frac{1}{T} \log \frac{F_0(q)}{S_0^{iq} \, \mathrm{e}^{r(iq-1)T}}, \tag{78}$$

we find that

$$D_0(q) = -iq \left(\psi(\sigma - \lambda) - \psi(-\lambda) \right) + \psi(iq\sigma - \lambda) - \psi(-\lambda) = \tilde{\psi}(iq\sigma) - iq \, \tilde{\psi}(\sigma), \tag{79}$$

where $\tilde{\psi}(\alpha) = \psi(\alpha - \lambda) - \psi(-\lambda)$. Without of loss of generality we can then set $\sigma = 1$ to get

$$D_0(q) = \tilde{\psi}(iq) - iq \, \tilde{\psi}(1). \tag{80}$$

This implies that $\tilde{\psi}(iq) = D_0(q) + iqb$ for some $b \in \mathbb{R}$. Now, $\psi(\alpha) = \tilde{\psi}(\alpha + \lambda) - \tilde{\psi}(\lambda)$. It follows that for some λ and b the Lévy exponent takes the form

$$\psi(iq) = D_0(iq + \lambda) - D_0(\lambda) + ibq. \tag{81}$$

Thus, we see that once we have been given a range of price data for imaginary power-payoff options, we can work out the Lévy exponent modulo a transformation of the form

$$\psi(iq) \rightarrow \psi(iq + \mu) - \psi(\mu) + icq, \tag{82}$$

where c and μ are constants.

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