Lévy-Vasicek Models and the Long-Bond Return Process

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The classical derivation of the well-known Vasicek model for interest rates is reformulated in terms of the associated pricing kernel. An advantage of the pricing kernel method is that it allows one to generalize the construction to the Lévy-Vasicek case, avoiding issues of market incompleteness. In the Lévy-Vasicek model the short rate is taken in the real-world measure to be a mean-reverting process with a general one-dimensional Lévy driver admitting exponential moments. Expressions are obtained for the Lévy-Vasicek bond prices and interest rates, along with a formula for the return on a unit investment in the long bond, defined by $L_t = \lim_{T \to \infty} P_{tT}/P_{0T}$, where P_{tT} is the price at time t of a T-maturity discount bond. We show that the pricing kernel of a Lévy-Vasicek model is uniformly integrable if and only if the long rate of interest is strictly positive.

I. PRICING KERNELS

The Vasicek model (Vasicek 1977) is one of the oldest and most well-studied models in the mathematical finance literature, and one might think that there is little that is new that can be said about it. But it turns out that there are some surprising features of the Vasicek model relating to the long rate of interest that are very suggestive when it comes to modelling long term interest rates in general.

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_t\}_{t \geq 0}$. Time 0 denotes the present. The probability measure $\mathbb{P}$ is the physical measure, and $\{\mathcal{F}_t\}$ represents the flow of market information. We introduce an appropriate unit of account, and for T and t such that $0 \leq t < T$ we let P_{tT} denote the value at time t of a discount bond that pays out one unit of account at maturity T. In what follows we use a pricing kernel method to construct the Vasicek model. This is not the way in which the Vasicek model is usually presented in the literature. Nevertheless, the pricing kernel approach is very effective. In particular, the pricing kernel formulation of the classical Vasicek model leads us directly to a construction of the corresponding Lévy-Vasicek model, extending results of Cairns (1999), Eberlein & Raible (1999), Norberg (2004), and others.

We begin with a few remarks about pricing kernels and then turn to the case of the Vasicek model. Let us recall briefly how pricing kernels work in the elementary geometric Brownian motion (GBM) model for asset prices. We fix a Brownian motion $\{W_t\}_{t \geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, and take it to be adapted to $\{\mathcal{F}_t\}$. The GBM model is characterized by the specification of a pricing kernel along with a collection of one or more so-called investment-grade assets. We assume for simplicity that the assets pay no dividends over the time horizon considered. The idea of an investment-grade asset is that it should offer a positive excess

rate of return above the interest rate. Ordinary stocks and bonds, for example, are in this sense investment-grade, whereas put options are not.

For the pricing kernel in the GBM model we assume that we have an expression of the form

$$\pi_t = e^{-rt} e^{-\lambda W_t - \frac{1}{2}\lambda^2 t}, \tag{1}$$

where r is the interest rate, and $\lambda > 0$ is a risk aversion parameter. We require that the product of the pricing kernel and the asset price should be a $\mathbb{P}$-martingale. Let us suppose that for some $\beta \geq -\lambda$ the product takes the form

$$\pi_t S_t = S_0 e^{\beta W_t - \frac{1}{2}\beta^2 t}, \tag{2}$$

where S_t denotes the value of the asset at time t. For a typical non-dividend-paying investment-grade asset in the GBM model we thus have

$$S_t = S_0 e^{(r+\lambda\sigma)t} e^{\sigma W_t - \frac{1}{2}\sigma^2 t}, \tag{3}$$

where $\sigma = \beta + \lambda$. The term $\lambda\sigma$ is called the risk premium or excess rate of return, and is positive under the assumptions we have made. The idea of a "pricing kernel" dates back to the 1970s and is used by Ross (1978). The alternative term "market kernel" is used by Garman (1976). Authors have employed a variety of terms for the same concept. Economists speak of the "marginal rate of substitution". The term "state price density" appears in Dothan & Williams (1978). The term "stochastic discount factor" is used in Cox & Martin (1983). The term "state price deflator" is used in Duffie (1992).

Pricing kernel models are discussed in detail in Cochrane (2005) and Hunt & Kennedy (2004). If the risky asset is a European-style derivative whose terminal payoff is H_T, then the value of the derivative at time $t < T$ is given by

$$H_t = \frac{1}{\pi_t} \mathbb{E}_t[\pi_T H_T]. \tag{4}$$

In particular, if the derivative pays one unit of account so that $H_T = 1$, then we recover the pricing formula for a discount bond, given by

$$P_{tT} = \frac{1}{\pi_t} \mathbb{E}_t[\pi_T]. \tag{5}$$

We refer to the process $\{n_t\}_{t \geq 0}$ defined by $n_t = 1/\pi_t$ as the "natural numeraire" (Flesaker & Hughston 1997), or "growth-optimal portfolio". It serves as a benchmark, relative to which other non-dividend-paying assets are martingales. As an example of derivative pricing in the GBM model, we consider the valuation of a digital put on the natural numeraire, with unit notional, strike κ, and maturity T. In this case we have

$$H_T = \mathbb{1}\left\{ n_T < \kappa \right\}, \tag{6}$$

where $\mathbb{1}\{\cdot\}$ is the indicator function. Using the pricing kernel (1), a straightforward calculation gives

$$H_0 = e^{-rT} N \left[\frac{\log\left(e^{-rT}\kappa\right) + \frac{1}{2}\lambda^2 T}{\lambda\sqrt{T}} \right], \tag{7}$$

where $N[\cdot]$ is the normal distribution function. We mention the example of a digital put on the natural numeraire because it turns out to be relevant later in our consideration of the uniform integrability of the pricing kernel. .

II. VASICEK PRICING KERNEL

We can extend the geometric Brownian motion model by keeping the risk-aversion level constant, but allowing the interest rate to be stochastic. The unit-initialized money market account is then given by

$$B_t = \exp \int_0^t r_s \mathrm{d}s, \tag{8}$$

and the pricing kernel is of the form

$$\pi_t = \exp \left[-\int_0^t r_s \mathrm{d}s - \lambda W_t - \frac{1}{2}\lambda^2 t \right]. \tag{9}$$

In the Vasicek model, the short rate process $\{r_t\}_{t \geq 0}$ is taken to be a mean-reverting process of the Ornstein-Uhlenbeck (OU) type, satisfying

$$\mathrm{d}r_t = k(\theta - r_t)\mathrm{d}t - \sigma \mathrm{d}W_t. \tag{10}$$

Here k, θ, and σ denote the mean reversion rate, the mean reversion level, and the absolute volatility of the short rate. Without loss of generality we can choose σ to be strictly positive. The initial value of the interest rate is r_0. The dynamical equation (10) can be solved by use of an integrating factor to give

$$r_t = \theta + (r_0 - \theta)\, \mathrm{e}^{-kt} - \sigma \int_0^t \mathrm{e}^{k(s-t)}\mathrm{d}W_s. \tag{11}$$

To obtain explicit formulae for the money market account and the pricing kernel in the Vasicek model we require an expression for the integrated short rate,

$$I_t = \int_0^t r_s \mathrm{d}s. \tag{12}$$

Substitution of (11) into (12) gives

$$I_t = \theta t + \frac{1}{k}\left(1 - \mathrm{e}^{-kt}\right)(r_0 - \theta) - \sigma \int_{s=0}^t \int_{u=0}^s \mathrm{e}^{k(u-s)}\mathrm{d}W_u \mathrm{d}s. \tag{13}$$

Now, by the Ito product rule we have

$$\mathrm{d}\left(\mathrm{e}^{-ks} \int_{u=0}^s \mathrm{e}^{ku}\mathrm{d}W_u \right) = \mathrm{d}W_s - k\mathrm{e}^{-ks} \int_{u=0}^s \mathrm{e}^{ku}\mathrm{d}W_u\, \mathrm{d}s. \tag{14}$$

Integrating each side of this equation and rearranging the result we obtain

$$\int_{s=0}^t \int_{u=0}^s \mathrm{e}^{k(u-s)}\mathrm{d}W_u \mathrm{d}s = \frac{1}{k}\int_0^t (1 - \mathrm{e}^{k(u-t)})\mathrm{d}W_u. \tag{15}$$

It follows that for the integrated short rate we have

$$I_t = \theta t + \frac{1}{k}\left(1 - \mathrm{e}^{-kt}\right)(r_0 - \theta) - \frac{\sigma}{k}\int_0^t (1 - \mathrm{e}^{k(u-t)})\mathrm{d}W_u, \tag{16}$$

and that the pricing kernel is given by

$$\pi_t = \exp\left[-\left(\theta + \frac{1}{2}\lambda^2\right)t - \frac{1}{k}\left(1 - e^{-kt}\right)(r_0 - \theta) + \int_0^t \left(\frac{\sigma}{k} - \lambda - \frac{\sigma}{k}e^{k(u-t)}\right)dW_u\right]. \quad (17)$$

A useful alternative expression for the integrated short rate can be obtained by integrating equation (10) to give

$$I_t = \theta t + \frac{1}{k}(r_0 - r_t) - \frac{\sigma}{k}W_t. \quad (18)$$

This expression can also be obtained by combining (11) and (16). It follows that the Vasicek money market account is given by

$$B_t = \exp\left[\theta t - \frac{\sigma}{k}W_t - \frac{1}{k}(r_t - r_0)\right], \quad (19)$$

and that the Vasicek pricing kernel can be expressed in the form

$$\pi_t = \exp\left[-\left(\theta + \frac{1}{2}\lambda^2\right)t + \left(\frac{\sigma}{k} - \lambda\right)W_t + \frac{1}{k}(r_t - r_0)\right]. \quad (20)$$

Note the appearance of W_t in the formulae for the money market account and the pricing kernel. Often it is said that the Vasicek model has a single state variable, the short rate. This statement is a little misleading. For while it is true, as we shall see shortly, that the price at time t of a T-maturity discount bond depends only on the state variable r_t insofar as its stochasticity is concerned, the money market account and the pricing kernel each depend at time t on a pair of state variables, namely, r_t and W_t. We take the view that to specify a financial model one needs to give the price processes of the basic assets of the model, together with the process for the pricing kernel. In the case of an interest rate model this means giving the processes for discount bonds of all maturities, the money market account, and the pricing kernel. Hence, the Vasicek model requires two state variables.

III. DISCOUNT BONDS

We proceed to derive an expression for P_{tT}. In the derivation we find it convenient to work with logarithms rather than exponentials. Thus, instead of (17) we write

$$\log \pi_t = -\left(\theta + \frac{1}{2}\lambda^2\right)t - \frac{1}{k}\left(1 - e^{-kt}\right)(r_0 - \theta) + \int_0^t \left(\frac{\sigma}{k} - \lambda - \frac{\sigma}{k}e^{k(u-t)}\right)dW_u. \quad (21)$$

It follows that

$$\log \pi_T = -\left(\theta + \frac{1}{2}\lambda^2\right)T - \frac{1}{k}\left(1 - e^{-kT}\right)(r_0 - \theta) + \int_0^T \left(\frac{\sigma}{k} - \lambda - \frac{\sigma}{k}e^{k(u-T)}\right)dW_u$$
$$+ \int_0^T \left(\frac{\sigma}{k} - \lambda - \frac{\sigma}{k}e^{k(u-T)}\right)dW_u, \quad (22)$$

and hence for $t < T$ we have

$$
\begin{aligned}
\mathbb{E}_t[\pi_T] = \exp&\left[-\left(\theta + \frac{1}{2}\lambda^2\right)T - \frac{1}{k}\left(1 - \mathrm{e}^{-kT}\right)(r_0 - \theta)\right.\\
&\left.+ \int_0^t \left(\frac{\sigma}{k} - \lambda - \frac{\sigma}{k}\mathrm{e}^{k(u-T)}\right)\mathrm{d}W_u\right]\\
&\times \mathbb{E}_t\left[\exp\left(\int_t^T \left(\frac{\sigma}{k} - \lambda - \frac{\sigma}{k}\mathrm{e}^{k(u-T)}\right)\mathrm{d}W_u\right)\right].
\end{aligned}
\tag{23}
$$

It is a standard result that for any measurable function $\{\alpha_t\}$ satisfying

$$
\int_t^T \alpha_u^2 \mathrm{d}u < \infty
\tag{24}
$$

we have

$$
\mathbb{E}_t\left[\exp\left(\int_t^T \alpha_u \mathrm{d}W_u\right)\right] = \exp\left(\frac{1}{2}\int_t^T \alpha_u^2 \mathrm{d}u\right).
\tag{25}
$$

As a consequence, we obtain

$$
\mathbb{E}_t \exp\left[\int_t^T \left(\frac{\sigma}{k} - \lambda - \frac{\sigma}{k}\mathrm{e}^{k(u-T)}\right)\mathrm{d}W_u\right] = \exp\left[\frac{1}{2}\int_t^T \left(\frac{\sigma}{k} - \lambda - \frac{\sigma}{k}\mathrm{e}^{k(u-T)}\right)^2 \mathrm{d}u\right].
\tag{26}
$$

Therefore, by (5) we have

$$
\begin{aligned}
\log P_{tT} = &-\left(\theta + \frac{1}{2}\lambda^2\right)(T - t) - \frac{1}{k}\left(\mathrm{e}^{-kt} - \mathrm{e}^{-kT}\right)(r_0 - \theta)\\
&+ \frac{1}{2}\int_t^T \left(\frac{\sigma}{k} - \lambda - \frac{\sigma}{k}\mathrm{e}^{k(u-T)}\right)^2 \mathrm{d}u + \frac{\sigma}{k}\left(1 - \mathrm{e}^{k(t-T)}\right)\int_0^t \mathrm{e}^{k(u-t)}\mathrm{d}W_u.
\end{aligned}
\tag{27}
$$

By use of (11), in the final term above we can write

$$
\sigma\int_0^t \mathrm{e}^{k(u-t)}\mathrm{d}W_u = \theta + (r_0 - \theta)\mathrm{e}^{-kt} - r_t.
\tag{28}
$$

Then the terms in (27) involving $r_0 - \theta$ cancel, and we are left with the following:

$$
\begin{aligned}
\log P_{tT} = &-\left(\theta + \frac{1}{2}\lambda^2\right)(T - t) + \frac{1}{2}\int_t^T \left(\frac{\sigma}{k} - \lambda - \frac{\sigma}{k}\mathrm{e}^{k(u-T)}\right)^2 \mathrm{d}u\\
&+ \frac{1}{k}\left(1 - \mathrm{e}^{k(t-T)}\right)(\theta - r_t).
\end{aligned}
\tag{29}
$$

Thus, we have isolated the dependence of P_{tT} on the state variable r_t. Now, for a, b constant we have

$$
\int_t^T \left(a - b\mathrm{e}^{ku}\right)^2 \mathrm{d}u = a^2(T - t) - \frac{2ab}{k}\left(\mathrm{e}^{kT} - \mathrm{e}^{kt}\right) + \frac{1}{2}\frac{b^2}{k}\left(\mathrm{e}^{2kT} - \mathrm{e}^{2kt}\right).
\tag{30}
$$

Hence,

$$\int_t^T \left(\frac{\sigma}{k} - \lambda - \frac{\sigma}{k}e^{k(u-T)}\right)^2 du = \left(\frac{\sigma}{k} - \lambda\right)^2 (T-t) - 2\left(\frac{\sigma^2}{k^3} - \frac{\lambda\sigma}{k^2}\right)\left(1 - e^{k(t-T)}\right)$$
$$+ \frac{1}{2}\frac{\sigma^2}{k^3}\left(1 - e^{2k(t-T)}\right). \tag{31}$$

Inserting this expression into (29), we see that the terms involving λ^2 cancel. After some simplification we therefore obtain the following expression for the value of a T-maturity discount bond:

$$P_{tT} = \exp\left[-R_\infty(T-t) + \frac{1}{k}\left(1 - e^{k(t-T)}\right)(R_\infty - r_t) - \frac{1}{4}\frac{\sigma^2}{k^3}\left(1 - e^{k(t-T)}\right)^2\right], \tag{32}$$

where

$$R_\infty = \theta + \frac{\lambda\sigma}{k} - \frac{1}{2}\frac{\sigma^2}{k^2}. \tag{33}$$

The prices of bonds in the Vasicek model have been known for forty years. Nevertheless, even seasoned practitioners may find it useful to see a derivation based entirely on pricing kernel methods, without the use of PDEs or measure change. The significance of R_∞ is that it represents the asymptotic bond yield, or "exponential long rate of interest", defined by

$$R_\infty = -\lim_{T\to\infty}\frac{1}{T-t}\log P_{tT}. \tag{34}$$

That R_∞ does not depend on t is characteristic of interest rate models for which the tail of the discount function is exponential, and is a manifestation of the so-called DIR theorem (Dybvig, Ingersoll & Ross 1996, Hubalek, Klein & Teichmann 2002, Goldammer & Schmock 2012, Kardaras & Platen 2012, Brody & Hughston 2016), according to which the long rate of interest can never fall in arbitrage-free term structure models.

IV.   UNIFORM INTEGRABILITY OF PRICING KERNEL

We proceed to remark on a feature of the Vasicek model that seems not to have been noted previously, namely, that $R_\infty > 0$ if and only if the pricing kernel is uniformly integrable (UI). Before we establish this fact and its generalization to the Lévy-Vasicek model, we propose to express the condition that the pricing kernel should be UI in financial terms. A collection $\mathcal{C}$ of random variables is said to be UI (see Williams 1991) if for every $\epsilon > 0$ there exists a $\delta \geq 0$ such that for all $X \in \mathcal{C}$ it holds that

$$\mathbb{E}[|X|\mathbb{1}\{|X| > \delta\}] < \epsilon. \tag{35}$$

An equivalent way of expressing the UI condition on $\mathcal{C}$ is

$$\lim_{\delta\to\infty}\sup_{X\in\mathcal{C}}\mathbb{E}[|X|\mathbb{1}\{|X| > \delta\}] = 0. \tag{36}$$

The limit on the left side of (36) exists since $\sup_{X \in \mathcal{C}} \mathbb{E}[|X|\mathbb{1}\{|X| > \delta\}]$ is decreasing in δ and bounded from below by zero. A random process $\{X_t\}_{t \geq 0}$ is then UI iff for every $\epsilon > 0$ there is a $\delta > 0$ such that $\mathbb{E}[|X_t|\mathbb{1}\{|X_t| > \delta\}] < \epsilon$ for all $t \geq 0$, or

$$\lim_{\delta \to \infty} \sup_t \mathbb{E}[|X_t|\,\mathbb{1}\{|X_t| > \delta\}] = 0. \tag{37}$$

In the case of a pricing kernel we have $\pi_t > 0$, so the UI condition is that for every $\epsilon > 0$ there should exist a $\delta \geq 0$ such that $\mathbb{E}[\pi_t \mathbb{1}\{\pi_t > \delta\}] < \epsilon$ for all $t \geq 0$, or

$$\lim_{\delta \to \infty} \sup_t \mathbb{E}[\pi_t\,\mathbb{1}\{\pi_t > \delta\}] = 0. \tag{38}$$

Alternatively, the UI condition can be imposed by requiring that for every $\epsilon > 0$ there should exist a $\kappa > 0$ such that

$$\mathbb{E}[\pi_t\,\mathbb{1}\{n_t < \kappa\}] < \epsilon \tag{39}$$

for all $t \geq 0$. Here $n_t = 1/\pi_t$, and we have set $\kappa = 1/\delta$. But the left side of (39) is the price at time 0 of a European-style digital put option on the natural numeraire with strike κ and maturity t. Thus we have shown:

Proposition 1 *A pricing kernel is uniformly integrable if and only if for any price level $\epsilon > 0$ there exists a strike $\kappa > 0$ such that the value of a digital put option on the natural numeraire is less than ϵ for all maturities.*

The class of interest rate models for which the pricing kernel has the UI property is in fact rather broad. For example, if the pricing kernel is a type-D potential (Hunt & Kennedy 2004, Rogers 1997, Rutkowski 1997), then it is UI (Meyer 1966). We proceed by establishing the following for the Vasicek model.

Proposition 2 *If $R_\infty > 0$ then $\{\pi_t\}$ is uniformly integrable.*

Proof. We shall use an $\mathcal{L}^p$ test. A collection $\mathcal{C}$ of random variables is said to be bounded in $\mathcal{L}^p$ if there exists a constant $\gamma > 0$ such that $\mathbb{E}[|X|^p] < \gamma$ for all $X \in \mathcal{C}$. Now, if $p > 1$ and $x \geq \delta > 0$ for $x, \delta \in \mathbb{R}$, then clearly $x \leq \delta^{1-p}x^p$. It follows that if $\mathcal{C}$ is bounded in $\mathcal{L}^p$ then for all $X \in \mathcal{C}$ it holds that

$$\mathbb{E}[|X|\mathbb{1}\{|X| > \delta\}] \leq \delta^{1-p}\mathbb{E}[|X^p|\mathbb{1}\{|X| > \delta\}] < \gamma\delta^{1-p}. \tag{40}$$

Therefore given any $\epsilon > 0$ if we set

$$\delta = (\epsilon/\gamma)^{1/(1-p)} \tag{41}$$

then we have constructed a δ such that (35) holds for all $X \in \mathcal{C}$. Thus, if a collection of random variables is bounded in $\mathcal{L}^p$ for some $p > 1$ then it is UI. Therefore, a sufficient condition for the pricing kernel to be UI is that there should exist a $p > 1$ and a $\gamma > 0$ such that $\mathbb{E}[\pi_t^p] < \gamma$ for all t. A calculation starting with (17) gives

$$\log \mathbb{E}[\pi_t^p] = -p\left[\theta + \frac{1}{2}\lambda^2 - p\frac{1}{2}\left(\frac{\sigma}{k} - \lambda\right)^2\right]t$$
$$+ \frac{p}{k}\left[\theta - r_0 - p\frac{\sigma}{k}\left(\frac{\sigma}{k} - \lambda\right)\right]\left(1 - \mathrm{e}^{-kt}\right) + \frac{p^2\sigma^2}{4k^3}\left(1 - \mathrm{e}^{-2kt}\right). \tag{42}$$

The second and third terms on the right are bounded, so our goal is to show that if $R_\infty > 0$ then there exists a value of $p > 1$ such that the coefficient of t in the first term on the right in (42) is less than or equal to zero. But if $R_\infty > 0$ then

$$\theta > \frac{1}{2}\frac{\sigma^2}{k^2} - \frac{\lambda\sigma}{k}. \tag{43}$$

Completing the square on the right, we get

$$\theta + \frac{1}{2}\lambda^2 > \frac{1}{2}\left(\frac{\sigma}{k} - \lambda\right)^2. \tag{44}$$

Therefore if we set

$$p = \frac{\theta + \frac{1}{2}\lambda^2}{\frac{1}{2}\left(\sigma k^{-1} - \lambda\right)^2}, \tag{45}$$

then $p > 1$ and the first term on the right side of (42) vanishes. This shows that if $R_\infty > 0$ then the $\mathcal{L}^p$ test is satisfied, and the pricing kernel is UI. $\square$

Proposition 3 *If $R_\infty < 0$ then $\{\pi_t\}$ is not uniformly integrable.*

Proof. If a collection $\mathcal{C}$ of random variables is UI then it is bounded in $\mathcal{L}^1$. For suppose that $\mathcal{C}$ is such that for every $\epsilon > 0$ there exists a $\delta \geq 0$ such that (35) holds for all $X \in \mathcal{C}$. Then there exists a constant δ_1 such that $\mathbb{E}[|X|\mathbb{1}\{|X| > \delta_1\}] < 1$ for all $X \in \mathcal{C}$, and therefore

$$\mathbb{E}[|X|] = \mathbb{E}[|X|\mathbb{1}\{|X| > \delta_1\}] + \mathbb{E}[|X|\mathbb{1}\{|X| \leq \delta_1\}] < 1 + \delta_1 \tag{46}$$

all $X \in \mathcal{C}$, and it follows that $\mathcal{C}$ is bounded in $\mathcal{L}^1$. Thus to establish the proposition it suffices to show that if $R_\infty < 0$ then $\{\pi_t\}$ is not bounded in $\mathcal{L}^1$. Keeping in mind that $\mathbb{E}[\pi_t] = P_{0t}$, we shall show that if $R_\infty < 0$ then for any choice of $\gamma > 0$ there exists a time t^* such that $P_{0t} > \gamma$ for all $t \geq t^*$. By virtue of (42), we have

$$P_{0t} = \exp\left[-R_\infty t + \frac{1}{k}\left(1 - \mathrm{e}^{-kt}\right)(R_\infty - r_0) - \frac{1}{4}\frac{\sigma^2}{k^3}\left(1 - \mathrm{e}^{-kt}\right)^2\right]. \tag{47}$$

It follows that

$$P_{0t} \geq \exp\left[-R_\infty t + \frac{1}{k}(R_\infty - r_0)\mathbb{1}\{R_\infty - r_0 \leq 0\} - \frac{1}{4}\frac{\sigma^2}{k^3}\right]. \tag{48}$$

Now suppose that $R_\infty < 0$ and define t^* by setting

$$-R_\infty t^* + \frac{1}{k}(R_\infty - r_0)\mathbb{1}\{R_\infty - r_0 \leq 0\} - \frac{1}{4}\frac{\sigma^2}{k^3} = \log\gamma. \tag{49}$$

Then for all $t > t^*$ we have $P_{0t} > \gamma$, which shows $\{\pi_t\}$ is not bounded in $\mathcal{L}^1$. $\square$

Remark. More generally, it can be shown that in any arbitrage-free interest rate model based on a pricing kernel with the property that the long exponential rate of interest exists and is negative the pricing kernel is not UI.

Proposition 4 *If $R_\infty = 0$ then $\{\pi_t\}$ is not uniformly integrable.*

Proof. The pricing kernel fails the $\mathcal{L}^p$ test if $R_\infty = 0$, so we cannot conclude that it is UI. On the other hand, the pricing kernel is bounded in $\mathcal{L}^1$ if $R_\infty = 0$, so we cannot conclude that it is *not* UI. Thus when $R_\infty = 0$ the simple tests give us no information and we need to look at the definition of uniform integrability and ask whether (38) holds. We shall demonstrate that if $R_\infty = 0$ then (38) does not hold, and therefore the pricing kernel is not UI. First, let us define

$$\alpha_{st} = \frac{\sigma}{k} - \lambda - \frac{\sigma}{k} \, e^{k(s-t)}. \tag{50}$$

Using (21) and (27) we can write

$$\pi_t = P_{0t} \exp\left(\int_0^t \alpha_{st} \, dW_s - \frac{1}{2} \int_0^t \alpha_{st}^2 \, ds \right). \tag{51}$$

Thus, for each value of t the pricing kernel is of the form

$$\pi_t = P_{0t} \exp\left(A_t Z - \frac{1}{2} A_t^2 \right), \tag{52}$$

where Z is normally distributed with mean zero and variance unity, and where we define A_t (which we take to be positive) by

$$A_t^2 = \int_0^t \alpha_{st}^2 \, ds. \tag{53}$$

It follows that

$$\mathbb{E}[\pi_t \mathbb{1}\{\pi_t > \delta\}] = P_{0t} \, \mathbb{E}\left[\exp\left(A_t Z - \frac{1}{2} A_t^2 \right) \mathbb{1}\left\{ Z > \frac{\log \delta - \log P_{0t} + \frac{1}{2} A_t^2}{A_t} \right\} \right]. \tag{54}$$

The expectation can be computed by standard techniques, leading to the following:

$$\mathbb{E}[\pi_t \mathbb{1}\{\pi_t > \delta\}] = P_{0t} \, N\left(\frac{\log P_{0t} + \frac{1}{2} A_t^2 - \log \delta}{A_t} \right). \tag{55}$$

Recall from (47) with $R_\infty = 0$ that

$$\log P_{0t} = -r_0 \frac{1}{k} \left(1 - e^{-kt} \right) - \frac{1}{4} \frac{\sigma^2}{k^3} \left(1 - e^{-kt} \right)^2, \tag{56}$$

which is bounded. We thus have

$$\sup_t \mathbb{E}[\pi_t \mathbb{1}(\pi_t > \delta)] \geq \exp\left[\inf_u \log P_{0u} \right] \sup_t N\left(\frac{\log P_{0t} + \frac{1}{2} A_t^2 - \log \delta}{A_t} \right)$$

$$\geq \exp\left[\inf_u \log P_{0u} \right] \sup_t N\left[\frac{\inf_u(\log P_{0u}) + \frac{1}{2} A_t^2 - \log \delta}{A_t} \right]. \tag{57}$$

It follows from

$$\inf_t \log P_{0t} = -\frac{r_0}{k}\mathbb{1}\{r_0 > 0\} - \frac{1}{4}\frac{\sigma^2}{k^3} \tag{58}$$

that

$$\sup_t \mathbb{E}[\pi_t \mathbb{1}\{\pi_t > \delta\}] \geq \exp\left[-\frac{r_0}{k}\mathbb{1}\{r_0 > 0\} - \frac{1}{4}\frac{\sigma^2}{k^3}\right]$$
$$\times \sup_t N\left[\frac{1}{A_t}\left(-\frac{r_0}{k}\mathbb{1}\{r_0 > 0\} - \frac{1}{4}\frac{\sigma^2}{k^3} + \tfrac{1}{2}A_t^2 - \log\delta\right)\right]. \tag{59}$$

Since $\lim_{t\to\infty} A_t = \infty$, the supremum on the right side is achieved in the limit as t approaches infinity. As a consequence, we have

$$\sup_t \mathbb{E}[\pi_t \mathbb{1}\{\pi_t > \delta\}] \geq \exp\left[-\frac{r_0}{k}\mathbb{1}\{r_0 > 0\} - \frac{1}{4}\frac{\sigma^2}{k^3}\right], \tag{60}$$

which implies that

$$\lim_{\delta\to\infty} \sup_t \mathbb{E}[\pi_t \mathbb{1}\{\pi_t > \delta\}] > 0, \tag{61}$$

and hence that the pricing kernel is not UI. $\square$

V. LONG-BOND RETURN PROCESS

The return at time t on an investment of one unit of account in the long bond is defined by the expression

$$L_t = \lim_{T\to\infty} \frac{P_{tT}}{P_{0T}}, \tag{62}$$

provided the limit exists (Flesaker & Hughston 1996). We refer to $\{L_t\}_{t\geq 0}$ as the long-bond return process. In the following, we consider the long-bond return process in the Vasicek model. We shall show that the limit exists and that it can be worked out explicitly. Using the formula for the discount bond price, we find that

$$\log \frac{P_{tT}}{P_{0T}} = R_\infty t + \frac{1}{k}(r_0 - r_t) - \frac{1}{k}(R_\infty - r_t)e^{-k(T-t)} + \frac{1}{k}(R_\infty - r_0)e^{-kT}$$
$$+ \frac{\sigma^2}{4k^3}\left(\left(1 - e^{-kT}\right)^2 - \left(1 - e^{-k(T-t)}\right)^2\right). \tag{63}$$

One sees immediately that

$$\lim_{T\to\infty} \log \frac{P_{tT}}{P_{0T}} = R_\infty t + \frac{1}{k}(r_0 - r_t). \tag{64}$$

It follows that the long-bond return process exists and is given by

$$L_t = \exp\left[R_\infty t + \frac{1}{k}(r_0 - r_t)\right]. \tag{65}$$

Recalling expression (20) for the pricing kernel in the Vasicek model and expression (33) for the long rate of interest, we deduce that the product of the pricing kernel and the long-bond return is given by

$$\pi_t L_t = \exp\left[\left(\frac{\sigma}{k} - \lambda\right) W_t - \frac{1}{2}\left(\frac{\sigma}{k} - \lambda\right)^2 t\right]. \tag{66}$$

This shows that the return on a unit investment in the long bond takes the form of a geometric Brownian motion asset with volatility σ/k and with a Vasicek-type integrated interest rate. More specifically, we have

$$L_t = \exp\left[\int_0^t \left(r_s + \frac{\lambda\sigma}{k}\right) \mathrm{d}s + \frac{\sigma}{k} W_t - \frac{1}{2}\left(\frac{\sigma}{k}\right)^2 t\right]. \tag{67}$$

The significance of the martingale $\{M_t\}_{t\geq 0}$ defined by $M_t = \pi_t L_t$ is that it acts as the change-of-measure density from the physical measure $\mathbb{P}$ to the so-called terminal measure (or long forward measure) introduced in Flesaker & Hughston (1996), which is the measure such that for any non-dividend-paying asset with price process $\{S_t\}$ the ratio $\{S_t/L_t\}$ is a martingale. To see this, recall that to change from $\mathbb{P}$ to the measure associated with a given numeraire $\{N_t\}$, the change-of-measure martingale is given by $\{\pi_t N_t\}$. For example, to change from $\mathbb{P}$ to the risk-neutral measure associated with the use of the money market account as numeraire, the change-of-measure martingale is $\{\pi_t B_t\}$. In the present context, the terminal measure agrees with $\mathbb{P}$ in the Vasicek model if and only if $M_t = 1$ for $t \geq 0$, which holds if and only if $\lambda = \sigma/k$. The condition that the terminal measure and the physical measure agree has been shown by Qin & Linetsky (2017) to be equivalent to the assumptions of the so-called recovery theorem of Ross (2015), which purports that under certain conditions it is possible to recover the physical measure from current option pricing data (Carr & Yu 2012, Borovička *et al* 2016). Thus, consistently with Qin & Linetsky (2017), one sees that in the Vasicek model under the Ross recovery hypothesis one can infer the market price of risk from the current price levels of options on discount bonds. This follows from the fact that such option prices depend on the ratio σ/k, which under the recover hypothesis is equal to the market price of risk. It should be emphasized, however, that there is no *a priori* reason to believe that the interest-rate market price of risk is equal to the volatility of the long-bond return process. In fact, we have:

Proposition 5 *In any arbitrage-free interest-rate model based on a Brownian filtration, Ross recovery holds if and only if the interest-rate market price of risk is equal to the volatility of the long-bond return process.*

Proof. We know that Ross recovery holds if and only if the terminal measure coincides with the physical measure, or equivalently $M_t = 1$ for all $t \geq 0$, which holds if and only if $L_t = 1/\pi_t$ for all $t \geq 0$, which in the case of interest-rate models based on a Brownian filtration holds if and only if the the interest-rate market price of risk agrees with the volatility of the return on the long bond. $\square$

Since there is no evidence that the interest-rate market price of risk is equal to the volatility of the long-bond return process, Proposition 5 shows that it is unlikely that Ross recovery will be observed in bond markets (Borovička *et al* 2016).

VI.   GEOMETRIC LÉVY MODELS

Are the foregoing conclusions, including those addressing the feasibility of Ross recovery and those relating the positivity of the long rate of interest to the uniform integrability of the pricing kernel, specific to markets based on Brownian filtrations? To investigate this question, we consider the more general case of markets based on Lévy filtrations. To begin, we review the pricing kernel approach to geometric Lévy models. The pricing kernel method has the advantage that it highlights the relations between risk, risk aversion, and return for models with price jumps (Brody *et al* 2012). Let $\{\xi_t\}$ be a Lévy process and $\lambda > 0$ the level of risk aversion. We assume that $\{\xi_t\}$ satisfies a moment condition of the form

$$\mathbb{E}\left[\exp(\alpha\xi_t)\right] < \infty \quad \text{for } t \geq 0 \text{ and } \alpha \in A, \tag{68}$$

for some interval $A \subset \mathbb{R}$ containing the origin as a proper subset. The pricing kernel of a geometric Lévy model, with constant interest rate r, is taken to be of the form

$$\pi_t = \mathrm{e}^{-rt}\,\mathrm{e}^{-\lambda\xi_t - \psi(-\lambda)t}, \tag{69}$$

where $-\lambda \in A$. Here $\{\psi(\alpha)\}_{\alpha \in A}$ is the so-called Lévy exponent, defined by

$$\mathbb{E}[\mathrm{e}^{\alpha\xi_t}] = \mathrm{e}^{\psi(\alpha)\,t}. \tag{70}$$

It is straightforward to check that the Lévy exponent is a strictly convex function. Since the product of the pricing kernel and the price $\{S_t\}$ of a non-dividend-paying asset is a $\mathbb{P}$-martingale, we generate a family of models for asset pricing (the so-called geometric Lévy models) by proposing that there should exist a constant β satisfying $\beta \geq -\lambda$ and $\beta + \lambda \in A$ such that

$$\pi_t S_t = S_0 \mathrm{e}^{\beta\xi_t - \psi(\beta)t}. \tag{71}$$

Writing $\sigma = \beta + \lambda$, we deduce that

$$S_t = S_0\,\mathrm{e}^{rt + R(\lambda,\sigma)t + \sigma\xi_t - \psi(\sigma)t}, \tag{72}$$

where the excess rate of return $R(\lambda, \sigma)$ is given by

$$R(\lambda, \sigma) = \psi(\sigma) + \psi(-\lambda) - \psi(\sigma - \lambda). \tag{73}$$

A short calculation shows that $R(\lambda, \sigma)$ is bilinear in λ and σ if and only if $\{\xi_t\}$ is a Brownian motion (Brody *et al* 2012). It follows that the interpretation of λ as a "market price of risk", which is valid for models based on a Brownian filtration, does not carry through to the general Lévy regime. Nevertheless, the notion of excess rate of return is well defined, and the convexity of the Lévy exponent implies that the excess rate of return is an increasing function of both λ and σ.

We draw attention to the fact that the value of the asset given by (72) does not depend on the drift of the Lévy process. More precisely, if one replaces ξ_t with $\xi_t + \mu t$ and $\psi(\alpha)$ with $\psi(\alpha) + \mu\alpha$, then one can easily check that S_t remains unchanged. Therefore without loss of generality we can set the drift of the Lévy process equal to zero. In that case we refer to $\{\xi_t\}$ as a compensated Lévy process. This implies that $\mathbb{E}[\xi_t] = 0$ and that $\{\xi_t\}$ is a martingale. For example, if $\{N_t\}$ is the standard Poisson process, with jump rate μ, then the associated compensated Lévy process is given by $\xi_t = N_t - \mu t$. With these conventions in mind we proceed to establish the following lemmas:

Lemma 1 *Let $\{\psi(\alpha)\}_{\alpha \in A}$ be the Lévy exponent of a compensated Lévy process $\{\xi_t\}$ that admits exponential moments for an interval $A \subset \mathbb{R}$ containing the origin as a proper subset. Then ψ is strictly positive on its domain, except at the origin, where it vanishes.*

Proof. Differentiating each side of (70) and setting $\alpha = 0$, we obtain $\mathbb{E}[\xi_t] = \psi'(0)t$ for all $t \geq 0$. Since $\{\psi(\alpha)\}$ is by assumption the Lévy exponent of a compensated Lévy process, it follows that $\psi'(0) = 0$. Hence, the curve $\psi : A \rightarrow \mathbb{R}$ defined by $\alpha \in A \rightarrow \psi(\alpha)$ has a horizontal tangent at the origin. Since ψ is strictly convex, and thus lies above any of its tangents except at the point where the tangent touches the curve, we conclude that ψ is strictly positive except at the origin. At the origin, we have $\psi(0) = 0$, which follows from the definition of the Lévy exponent. $\qquad\square$

Lemma 2 *Let $A \subset \mathbb{R}$ be an interval containing the origin as a proper subset, and let $f : A \rightarrow \mathbb{R}$ be a nonnegative strictly convex function, differentiable on A and vanishing at 0. Then it holds that $x f'(x) > f(x)$ for all $x \in A$ except at $x = 0$.*

Proof. Let $x \in A$. If $x > 0$, then by the mean value theorem there exists $y \in (0, x)$ such that $f(x) = x f'(y)$. Since f takes its minimum at the origin and is strictly convex, it follows that $f'(y) < f(x)$. Therefore, $f(x) < x f'(x)$, as required. On the other hand, if $x < 0$, then the mean value theorem says that there exists $y \in (x, 0)$ such that $f(x) = x f'(y)$. But since f is strictly convex with a minimum at the origin, it follows that $0 < f'(x) < f'(y)$, and thus $x f'(y) < x f'(x)$, since $x < 0$. Therefore, $f(x) < x f'(x)$. $\qquad\square$

We also have the following, which extends Lemma 3.1 of Eberlein & Raible (1999) by allowing the interval A to be asymmetric about the origin.

Lemma 3 *Let $\{\xi_s\}$ be a Lévy process such that $\mathbb{E}[\exp(\alpha\xi_t)] < \infty$ for $t \geq 0$ and $\alpha \in A \subset \mathbb{R}$ for some interval $A = [-(1+\epsilon)M, (1+\epsilon)N]$ where M, N and ϵ are strictly positive constants. Let the function $\alpha : \mathbb{R} \rightarrow A' = [-M, N]$ be left-continuous with right limits. Then it holds that*

$$\mathbb{E}\left[\exp\left(\int_0^t \alpha_s \mathrm{d}\xi_s\right)\right] = \exp\left(\int_0^t \psi(\alpha_s)\mathrm{d}s\right). \tag{74}$$

VII. CONSTRUCTION OF LÉVY-VASICEK MODEL

Going forward, we investigate properties of the pricing kernel in the Lévy analogue of the Vasicek model. We are interested in a short rate model of the Ornstein-Uhlenbeck type, driven by a Lévy process. Such models have been investigated by Norberg (2004). Remarkably, the condition $\lambda = \sigma/k$ for Ross recovery in the Vasicek model is unchanged in its Lévy-Vasicek counterpart.

The pricing kernel method allows us to work out the details of the general Lévy-Vasicek model in the $\mathbb{P}$-measure. We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and introduce a one-dimensional Lévy process $\{\xi_t\}_{t \geq 0}$. We assume that (68) holds for some $A \subset \mathbb{R}$ containing the origin as a proper subset, and we write $\psi(\alpha)$ for the Lévy exponent, defined for $\alpha \in A$. The pricing kernel in the Lévy-Vasicek model takes the form

$$\pi_t = \exp\left[-\int_0^t r_s\,\mathrm{d}s - \lambda\xi_t - \psi(-\lambda)\,t\right], \tag{75}$$

where the short rate is a Lévy-Ornstein-Uhlenbeck process satisfying a dynamical equation of the form

$$\mathrm{d}r_t = k(\theta - r_t)\,\mathrm{d}t - \sigma\mathrm{d}\xi_t. \tag{76}$$

The parameters of the model here have essentially the same interpretation as those of the classical Vasicek model. Without loss of generality we can set the drift of the Lévy process equal to zero by absorbing any drift into the definition of the mean-reversion level. Thus in what follows we assume that $\{\xi_t\}$ is a compensated Lévy process. We find that the short rate is given by

$$r_t = \theta + (r_0 - \theta)\mathrm{e}^{-kt} - \sigma\int_0^t \mathrm{e}^{k(s-t)}\mathrm{d}\xi_s. \tag{77}$$

The integrated short rate can be worked out by a calculation that parallels that of the classical Vasicek model, with the following result:

$$\int_0^t r_s\mathrm{d}s = \theta t + \frac{1}{k}(r_0 - r_t) - \frac{\sigma}{k}\xi_t\,. \tag{78}$$

It follows that the pricing kernel in the Lévy-Vasicek model can be written as

$$\pi_t = \exp\left[-(\theta + \psi(-\lambda))t + \left(\frac{\sigma}{k} - \lambda\right)\xi_t - \frac{1}{k}(r_0 - r_t)\right]. \tag{79}$$

Alternatively, if we insert the expression for r_t given in (77) then we obtain

$$\pi_t = \exp\left[-(\theta + \psi(-\lambda))\,t - \frac{1}{k}\left(1 - \mathrm{e}^{-kt}\right)(r_0 - \theta) + \int_0^t \left(\frac{\sigma}{k} - \lambda - \frac{\sigma}{k}\mathrm{e}^{k(u-t)}\right)\mathrm{d}\xi_u\right]. \tag{80}$$

VIII.   DISCOUNT BONDS IN LÉVY-VASICEK MODEL

We proceed to obtain an expression for the price of a discount bond. By (80) we have

$$\log\pi_T = -(\theta + \psi(-\lambda))\,t - \frac{1}{k}\left(1 - \mathrm{e}^{-kT}\right)(r_0 - \theta) + \int_0^T \left(\frac{\sigma}{k} - \lambda - \frac{\sigma}{k}\mathrm{e}^{k(u-T)}\right)\mathrm{d}\xi_u, \tag{81}$$

from which we deduce that

$$\log\pi_T = -(\theta + \psi(-\lambda))\,T - \frac{1}{k}\left(1 - \mathrm{e}^{-kT}\right)(r_0 - \theta) + \int_0^t \left(\frac{\sigma}{k} - \lambda - \frac{\sigma}{k}\mathrm{e}^{k(u-T)}\right)\mathrm{d}\xi_u$$
$$+ \int_t^T \left(\frac{\sigma}{k} - \lambda - \frac{\sigma}{k}\mathrm{e}^{k(u-T)}\right)\mathrm{d}\xi_u. \tag{82}$$

Hence,

$$\mathbb{E}_t[\pi_T] = \exp\left[-(\theta + \psi(-\lambda))\,T - \frac{1}{k}\left(1 - \mathrm{e}^{-kT}\right)(r_0 - \theta)\right.$$
$$\left.+ \int_0^t \left(\frac{\sigma}{k} - \lambda - \frac{\sigma}{k}\mathrm{e}^{k(u-T)}\right)\mathrm{d}\xi_u\right]$$
$$\times \mathbb{E}_t\left[\exp\left(\int_t^T \left(\frac{\sigma}{k} - \lambda - \frac{\sigma}{k}\mathrm{e}^{k(u-T)}\right)\mathrm{d}\xi_u\right)\right]. \tag{83}$$

A sufficient condition for the use of Lemma 3 to work out the conditional expectation on the right side of this expression is that the domain in $\mathbb{R}$ upon which the Lévy exponent is defined includes a subset of the form $[-(1+\epsilon)\lambda, (1+\epsilon)(\sigma k^{-1} - \lambda)]$ if $\sigma k^{-1} - \lambda > 0$, for some $\epsilon > 0$, or a subset of the form $[-(1+\epsilon)\lambda, 0]$ if $\sigma k^{-1} - \lambda < 0$. Then it follows from Lemma 3 that

$$\mathbb{E}_t \exp\left[\int_t^T \left(\frac{\sigma}{k} - \lambda - \frac{\sigma}{k}e^{k(u-T)}\right) d\xi_u\right] = \exp\left[\int_t^T \psi\left(\frac{\sigma}{k} - \lambda - \frac{\sigma}{k}e^{k(u-T)}\right) du\right]. \qquad (84)$$

Therefore, by expression (5) for the discount bond we obtain

$$\log P_{tT} = -\left(\theta + \psi(-\lambda)\right)(T-t) - \frac{1}{k}\left(e^{-kt} - e^{-kT}\right)(r_0 - \theta)$$
$$+ \int_t^T \psi\left(\frac{\sigma}{k} - \lambda - \frac{\sigma}{k}e^{k(u-T)}\right) du + \frac{\sigma}{k}\left(1 - e^{k(t-T)}\right) \int_0^t e^{k(u-t)} d\xi_u. \qquad (85)$$

As a consequence of (77) we can write

$$\sigma \int_0^t e^{k(u-t)} d\xi_u = \theta + (r_0 - \theta)e^{-kt} - r_t. \qquad (86)$$

Thus, the terms involving $r_0 - \theta$ in (85) cancel, and we are left with the following expression for the price of a T-maturity discount bond:

$$P_{tT} = \exp\left[-\left(\theta + \psi(\lambda)\right)(T-t) + \int_t^T \psi(\alpha_{uT}) du + \frac{1}{k}\left(1 - e^{k(t-T)}\right)(\theta - r_t)\right], \qquad (87)$$

where for $0 \le s \le t$ we set

$$\alpha_{ut} = \frac{\sigma}{k} - \lambda - \frac{\sigma}{k}e^{k(u-t)}. \qquad (88)$$

To investigate the asymptotic bond yield, or exponential long rate, first we show

$$\lim_{T\to\infty} \frac{1}{T-t} \int_t^T \psi\left(\alpha_{sT}\right) ds = \psi\left(\frac{\sigma}{k} - \lambda\right). \qquad (89)$$

We note that the derivative of the numerator with respect to T is given by

$$\frac{d}{dT} \int_t^T \psi(\alpha_{sT}) ds = \psi(\alpha_{TT}) + \sigma \int_t^T \psi'(\alpha_{sT})e^{k(s-T)} ds$$
$$= \psi(\alpha_{TT}) - \int_t^T \frac{d}{ds}\psi(\alpha_{sT}) ds$$
$$= \psi(\alpha_{tT}). \qquad (90)$$

Thus, applying l'Hospital's rule we obtain

$$\lim_{T\to\infty} \frac{1}{T-t} \int_t^T \psi\left(\alpha_{sT}\right) ds = \lim_{T\to\infty} \psi(\alpha_{tT}) = \psi\left(\frac{\sigma}{k} - \lambda\right), \qquad (91)$$

establishing (89). One sees that in the Lévy-Vasicek model, as in the Brownian case, the long rate does not depend on t, and we have the following:

$$R_\infty = -\lim_{T\to\infty} \frac{1}{T-t} \log P_{tT} = \theta + \psi(-\lambda) - \psi\left(\frac{\sigma}{k} - \lambda\right). \qquad (92)$$

IX.   UNIFORM INTEGRABILITY IN LÉVY-VASICEK MODEL

A natural question that emerges is whether and for what choice of parameters the pricing kernel in a Lévy-Vasicek model is UI. For this purpose, it will be useful to express the pricing kernel in the form

$$\pi_t = \exp\left(-(\theta + \psi(-\lambda))t - \frac{1}{k}(r_0 - \theta)(1 - \mathrm{e}^{-kt}) + \int_0^t \alpha_{st}\,\mathrm{d}\xi_s\right), \tag{93}$$

or equivalently

$$\pi_t = \exp\left(-R_\infty t - \psi\left(\frac{\sigma}{k} - \lambda\right)t - \frac{1}{k}(r_0 - \theta)(1 - \mathrm{e}^{-kt}) + \int_0^t \alpha_{st}\,\mathrm{d}\xi_s\right), \tag{94}$$

where we recall the definition (88) for α_{st}. It follows that

$$\log \mathbb{E}[\pi_t] = -R_\infty t - \frac{1}{k}(r_0 - \theta)(1 - \mathrm{e}^{-kt}) + \int_0^t \psi\left(\alpha_{st}\right)\mathrm{d}s - \psi\left(\frac{\sigma}{k} - \lambda\right)t. \tag{95}$$

Let us adopt the following conventions. Given a pair of functions $f : \mathbb{R}^+ \to \mathbb{R}$ and $g : \mathbb{R}^+ \to \mathbb{R}\backslash\{0\}$, we say that f is $O(g)$ for large t if

$$\limsup_{t\to\infty} \left|\frac{f(t)}{g(t)}\right| < \infty, \tag{96}$$

and we say that f is $o(g)$ for large t if

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} = 0. \tag{97}$$

With reference to the integral appearing on the right side of (95), we shall show

$$\int_0^t \psi\left(\alpha_{st}\right)\mathrm{d}s = \psi\left(\frac{\sigma}{k} - \lambda\right)t + O(1) \tag{98}$$

for large t. We note that

$$\left|\int_0^t \left(\psi\left(\alpha_{st}\right) - \psi\left(\frac{\sigma}{k} - \lambda\right)\right)\mathrm{d}s\right| \le \int_0^t \left|\psi\left(\alpha_{st}\right) - \psi\left(\frac{\sigma}{k} - \lambda\right)\right|\mathrm{d}s. \tag{99}$$

But it follows from the mean value theorem that, for a fixed value of s, there exists a ρ in the open interval $(-\sigma k^{-1}\mathrm{e}^{k(s-t)}, 0)$ such that

$$\psi\left(\frac{\sigma}{k} - \lambda\right) = \psi(\alpha_{st}) + \frac{\sigma}{k}\mathrm{e}^{k(s-t)}\psi'\left(\frac{\sigma}{k} - \lambda + \rho\right). \tag{100}$$

Hence,

$$\left|\psi\left(\alpha_{st}\right) - \psi\left(\frac{\sigma}{k} - \lambda\right)\right| = \frac{\sigma}{k}\mathrm{e}^{k(s-t)}\left|\psi'\left(\frac{\sigma}{k} - \lambda + \rho\right)\right|. \tag{101}$$

Recall from Lemma 1 that ψ is a nonnegative strictly convex function taking its minimum value at zero. Thus, over the relevant range of ρ, the maximum of $|\psi'(\sigma/k - \lambda + \rho)|$ is taken,

depending on the value of s, either at the left boundary $\rho = -(\sigma/k)\mathrm{e}^{k(s-t)}$ or at the right boundary $\rho = 0$. More precisely, there exists a $t' \in [0, t]$ such that as s varies, the maximum is achieved at the right boundary whenever $s \in (0, t')$, and at the left boundary whenever $s \in (t', t)$. Thus, when $s \in (0, t')$ we have

$$\left| \psi\left(\alpha_{st}\right) - \psi\left(\frac{\sigma}{k} - \lambda\right) \right| \leq \frac{\sigma}{k}\mathrm{e}^{k(s-t)} \left| \psi'\left(\frac{\sigma}{k} - \lambda\right) \right|, \tag{102}$$

and when $s \in (t', t)$ we have

$$\left| \psi\left(\alpha_{st}\right) - \psi\left(\frac{\sigma}{k} - \lambda\right) \right| \leq \frac{\sigma}{k}\mathrm{e}^{k(s-t)} \left| \psi'\left(\frac{\sigma}{k} - \lambda - \frac{\sigma}{k}\mathrm{e}^{k(s-t)}\right) \right| \leq \frac{\sigma}{k}\mathrm{e}^{k(s-t)} |\psi'(-\lambda)|. \tag{103}$$

In the last step here we have made use of the fact that since ψ is convex, $|\psi'|$ is decreasing on the negative half line. Reverting to the right side of (99), we have

$$\int_0^t \left| \psi\left(\alpha_{st}\right) - \psi\left(\frac{\sigma}{k} - \lambda\right) \right| \mathrm{d}s$$

$$= \int_0^{t'} \left| \psi\left(\alpha_{st}\right) - \psi\left(\frac{\sigma}{k} - \lambda\right) \right| \mathrm{d}s + \int_{t'}^t \left| \psi\left(\alpha_{st}\right) - \psi\left(\frac{\sigma}{k} - \lambda\right) \right| \mathrm{d}s$$

$$\leq \frac{\sigma}{k}\psi'\left(\frac{\sigma}{k} - \lambda\right)\int_0^{t'} \mathrm{e}^{k(s-t)}\mathrm{d}s + \frac{\sigma}{k}|\psi'(-\lambda)|\int_{t'}^t \mathrm{e}^{k(s-t)}\mathrm{d}s$$

$$= \frac{\sigma}{k^2}\psi'\left(\frac{\sigma}{k} - \lambda\right)\mathrm{e}^{-kt}\left(\mathrm{e}^{kt'} - 1\right) + \frac{\sigma}{k^2}|\psi'(-\lambda)|\left(1 - \mathrm{e}^{-k(t-t')}\right)$$

$$\leq \frac{\sigma}{k^2}\psi'\left(\frac{\sigma}{k} - \lambda\right) + \frac{\sigma}{k^2}|\psi'(-\lambda)|, \tag{104}$$

and hence

$$\left| \int_0^t \psi(\alpha_{st})\,\mathrm{d}s - \psi\left(\frac{\sigma}{k} - \lambda\right)t \right| \leq \frac{\sigma}{k^2}\psi'\left(\frac{\sigma}{k} - \lambda\right) + \frac{\sigma}{k^2}|\psi'(-\lambda)|. \tag{105}$$

It follows that

$$\limsup_{t\to\infty}\left| \int_0^t \psi(\alpha_{st})\,\mathrm{d}s - \psi\left(\frac{\sigma}{k} - \lambda\right)t \right| < \infty, \tag{106}$$

which establishes (98).

With these preparations at hand, we are ready to return to our considerations of the uniform integrability of the pricing kernel.

Proposition 6 *If $R_\infty > 0$ then $\{\pi_t\}$ is uniformly integrable.*

Proof. We shall use an $\mathcal{L}^p$ test. Specifically, we show that if $R_\infty > 0$ then there exists a $p > 1$ such that $\sup_t \mathbb{E}[\pi_t^p] < \infty$. We claim that it suffices to prove that if $R_\infty > 0$ then $\lim_{t\to\infty} \mathbb{E}[\pi_t^p] = 0$ for some $p > 1$. To see this, note that if $\lim_{t\to\infty} \mathbb{E}[\pi_t^p] = 0$ holds for some $p > 1$, there exist positive constants C and T such that $\mathbb{E}[\pi_t^p] < C$ for all $t \geq T$. But then

$$\sup_t \mathbb{E}[\pi_t^p] \leq \max\left(C, \sup_{t\leq T} \mathbb{E}[\pi_t^p]\right). \tag{107}$$

Since continuous functions are bounded on compact intervals, we see that $\sup_{t \leq T} \mathbb{E}[\pi_t^p]$ is bounded, and thus $\sup_t \mathbb{E}[\pi_t^p] < \infty$, as required. Let us therefore show that $\lim_{t \to \infty} \mathbb{E}[\pi_t^p] = 0$ for some $p > 1$. It follows from (94) that

$$\log \mathbb{E}[\pi_t^p] = -pR_\infty t - \frac{p}{k}(r_0 - \theta)(1 - \mathrm{e}^{-kt}) + \int_0^t \psi(p\alpha_{st})\,\mathrm{d}s - p\psi\left(\frac{\sigma}{k} - \lambda\right)t, \tag{108}$$

and we can use (98) to see that

$$\begin{aligned}\log \mathbb{E}[\pi_t^p] &= -pR_\infty t - \frac{p}{k}(r_0 - \theta)(1 - \mathrm{e}^{-kt}) \\ &\quad + \left(\psi\left(p\left(\frac{\sigma}{k} - \lambda\right)\right) - p\psi\left(\frac{\sigma}{k} - \lambda\right)\right)t + O(1)\end{aligned} \tag{109}$$

for large t. Since ψ is continuous at $\sigma/k - \lambda$, we observe that the quantity

$$\left|\left(\psi\left(p\left(\frac{\sigma}{k} - \lambda\right)\right) - p\psi\left(\frac{\sigma}{k} - \lambda\right)\right)\right| \tag{110}$$

can be made arbitrarily small as p approaches unity from above. Keeping in mind that $R_\infty > 0$, we thus see that there exists a $p > 1$ such that

$$-R_\infty + \left(\psi\left(p\left(\frac{\sigma}{k} - \lambda\right)\right) - p\psi\left(\frac{\sigma}{k} - \lambda\right)\right) < 0, \tag{111}$$

and hence $\lim_{t \to \infty} \mathbb{E}[\pi_t^p] = 0$, which is the required result. □

Proposition 7 *If $R_\infty < 0$ then $\{\pi_t\}$ is not uniformly integrable.*

Proof. We shall show that $\{\pi_t\}$ is not bounded in $\mathcal{L}^1$, which implies that $\{\pi_t\}$ is not UI. It follows from (95) and (98) that for large t one has

$$\log \mathbb{E}[\pi_t] = -R_\infty t - \frac{1}{k}(r_0 - \theta)(1 - \mathrm{e}^{-kt}) + O(1). \tag{112}$$

If $R_\infty < 0$, we have

$$\lim_{t \to \infty} \mathbb{E}[\pi_t] = \infty. \tag{113}$$

Thus, the pricing kernel is not bounded in $\mathcal{L}^1$. □

Proposition 8 *If $R_\infty = 0$ then $\{\pi_t\}$ is not uniformly integrable.*

Proof. We recall that the pricing kernel is UI if and only if (38) holds. Using (87) and (93) we have

$$\pi_t = P_{0t} \exp\left(\int_0^t \alpha_{st}\,\mathrm{d}\xi_s - \int_0^t \psi(\alpha_{st})\,\mathrm{d}s\right). \tag{114}$$

As a consequence, we see that

$$
\begin{aligned}
&\mathbb{E}[\pi_t \mathbb{1}\{\pi_t > \delta\}] \\
&\qquad = P_{0t}\mathbb{E}\left[\exp\left(\int_0^t \alpha_{st}\,\mathrm{d}\xi_s - \int_0^t \psi(\alpha_{st})\,\mathrm{d}s\right)\mathbb{1}\left\{\int_0^t \alpha_{st}\,\mathrm{d}\xi_s > B(t,\delta)\right\}\right],
\end{aligned}
$$

(115)

where we define

$$
B(t,\delta) = \log\delta - \log P_{0t} + \int_0^t \psi(\alpha_{st})\,\mathrm{d}s.
$$

(116)

Equivalently, by (87) we have

$$
B(t,\delta) = \log\delta + R_\infty t + \frac{1}{k}(r_0 - \theta)\left(1 - \mathrm{e}^{-kt}\right) + \psi\left(\frac{\sigma}{k} - \lambda\right)t.
$$

(117)

We introduce a new measure $\mathbb{P}^*$ on $\mathcal{F}_t$ by setting

$$
\mathbb{P}^*(A) = \mathbb{E}\left[\exp\left(\int_0^t \alpha_{st}\,\mathrm{d}\xi_s - \int_0^t \psi(\alpha_{st})\,\mathrm{d}s\right)\mathbb{1}\{A\}\right]
$$

(118)

for $A \in \mathcal{F}_t$. Writing $\mathbb{E}^*$ for expectation under $\mathbb{P}^*$ we then have

$$
\mathbb{E}[\pi_t \mathbb{1}\{\pi_t > \delta\}] = P_{0t}\,\mathbb{E}^*\left[\mathbb{1}\left\{\int_0^t \alpha_{st}\,\mathrm{d}\xi_s > B(t,\delta)\right\}\right].
$$

(119)

Let us introduce a positive constant ω with units of inverse time, making ωt dimensionless. Thus ω is a fixed "rate". To proceed, we need the following results regarding the mean and variance of the random variable $\int_0^t \alpha_{st}\mathrm{d}\xi_s$ under $\mathbb{P}^*$. If $R_\infty = 0$, then for large t we have

$$
\mathbb{E}^*\left[\int_0^t \alpha_{st}\,\mathrm{d}\xi_s\right] - B(t,\omega t) = Ct + o(t),
$$

(120)

where

$$
C = \left(\frac{\sigma}{k} - \lambda\right)\psi'\left(\frac{\sigma}{k} - \lambda\right) - \psi\left(\frac{\sigma}{k} - \lambda\right)
$$

(121)

is a positive constant, and

$$
\mathrm{Var}^*\left[\int_0^t \alpha_{st}\,\mathrm{d}\xi_s\right] = O(t).
$$

(122)

To see (120), (121), and (122), note that for u sufficiently close to zero we have

$$
\mathbb{E}^*\left[\exp\left(u\int_0^t \alpha_{st}\,\mathrm{d}\xi_s\right)\right] = \exp\left(\int_0^t \left(\psi\left((1+u)\alpha_{st}\right) - \psi(\alpha_{st})\right)\mathrm{d}s\right).
$$

(123)

Taking a derivative with respect to u on both sides and setting $u = 0$ gives

$$
\mathbb{E}^*\left[\int_0^t \alpha_{st}\,\mathrm{d}\xi_s\right] = \int_0^t \psi'(\alpha_{st})\alpha_{st}\,\mathrm{d}s.
$$

(124)

A calculation then shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}^*\left[\int_0^t \alpha_{st}\,\mathrm{d}\xi_s\right] = \alpha_{0t}\psi'(\alpha_{0t}), \tag{125}$$

and thus, by l'Hospital's rule,

$$\lim_{t\to\infty}\frac{1}{t}\mathbb{E}^*\left[\int_0^t \alpha_{st}\,\mathrm{d}\xi_s\right] = \lim_{t\to\infty}\alpha_{0t}\,\psi'(\alpha_{0t}) = \left(\frac{\sigma}{k}-\lambda\right)\psi'\left(\frac{\sigma}{k}-\lambda\right). \tag{126}$$

It follows that

$$\mathbb{E}^*\left[\int_0^t \alpha_{st}\,\mathrm{d}\xi_s\right] = \left(\frac{\sigma}{k}-\lambda\right)\psi'\left(\frac{\sigma}{k}-\lambda\right)t + o(t) \tag{127}$$

for large t. Recall from (117) that $B(t,\delta)$ grows to leading order like $\psi(\sigma/k - \lambda)t$ when $R_\infty = 0$. This remains the case if we replace δ by ωt, since the growth of $\log\omega t$ is slower than linear growth. That is,

$$\mathbb{E}^*\left[\int_0^t \alpha_{st}\,\mathrm{d}\xi_s\right] - B(t,\omega t) = \left[\left(\frac{\sigma}{k}-\lambda\right)\psi'\left(\frac{\sigma}{k}-\lambda\right) - \psi\left(\frac{\sigma}{k}-\lambda\right)\right]t + o(t) \tag{128}$$

for large t. The positivity of the coefficient of t follows from Lemma 2. We have thus arrived at (120) and (121). It remains to show equation (122). Taking the second derivative of (123) with respect to u and setting $u = 0$, we find

$$\mathbb{E}^*\left[\left(\int_0^t \alpha_{st}\,\mathrm{d}\xi_s\right)^2\right] = \left(\int_0^t \psi'(\alpha_{st})\alpha_{st}\,\mathrm{d}s\right)^2 + \int_0^t \psi''(\alpha_{st})\alpha_{st}^2\,\mathrm{d}s. \tag{129}$$

Using (125), we then obtain

$$\mathrm{Var}^*\left[\int_0^t \alpha_{st}\,\mathrm{d}\xi_s\right] = \mathbb{E}^*\left[\left(\int_0^t \alpha_{st}\,\mathrm{d}\xi_s\right)^2\right] - \left(\mathbb{E}^*\left[\int_0^t \alpha_{st}\,\mathrm{d}\xi_s\right]\right)^2$$

$$= \int_0^t \psi''(\alpha_{st})\alpha_{st}^2\,\mathrm{d}s. \tag{130}$$

The limit

$$\lim_{t\to\infty}\frac{1}{t}\mathrm{Var}^*\left[\int_0^t \alpha_{st}\,\mathrm{d}\xi_s\right] = \lim_{t\to\infty}\frac{1}{t}\int_0^t \psi''(\alpha_{st})\alpha_{st}^2\,\mathrm{d}s \tag{131}$$

is finite, which can be seen using l'Hospital's rule. Hence, (122) follows. We set

$$c(t) = \mathbb{E}^*\left[\int_0^t \alpha_{st}\,\mathrm{d}\xi_s\right] - B(t,\omega t), \tag{132}$$

and recall from (120) that $c(t)$ grows linearly for large t. Let t be large enough so that $c(t) > 0$. We recall that if X is a random variable such that $\mathrm{Var}\,[X] < \infty$, then for any constant $c > 0$ we have the Chebyshev inequality

$$\mathbb{P}\left[|X - \mathbb{E}[X]| \geq c\right] \leq \frac{1}{c^2}\mathrm{Var}\,[X]. \tag{133}$$

In the present context it follows that

$$\mathbb{P}^* \left[\left| \int_0^t \alpha_{st} d\xi_s - \mathbb{E}^* \left[\int_0^t \alpha_{st} d\xi_s \right] \right| \geq c(t) \right] \leq \frac{1}{c(t)^2} \text{Var}^* \left[\int_0^t \alpha_{st} d\xi_s \right]. \tag{134}$$

Since both $c(t)$ and the variance grow linearly in t by (120)–(122), we see that

$$\lim_{t \to \infty} \mathbb{P}^* \left[\left| \int_0^t \alpha_{st} d\xi_s - \mathbb{E}^* \left[\int_0^t \alpha_{st} d\xi_s \right] \right| \geq c(t) \right] = 0. \tag{135}$$

For t large enough so that $c(t) > 0$, we have

$$\begin{aligned}
\mathbb{E}^* \left[\mathbb{1} \left\{ \int_0^t \alpha_{st} d\xi_s > B(t, \omega t) \right\} \right] &= \mathbb{P}^* \left[\int_0^t \alpha_{st} d\xi_s > B(t, \omega t) \right] \\
&\geq \mathbb{P}^* \left[\left| \int_0^t \alpha_{st} d\xi_s - \mathbb{E}^* \left[\int_0^t \alpha_{st} d\xi_s \right] \right| < c(t) \right] \\
&= 1 - \mathbb{P}^* \left[\left| \int_0^t \alpha_{st} d\xi_s - \mathbb{E}^* \left[\int_0^t \alpha_{st} d\xi_s \right] \right| \geq c(t) \right].
\end{aligned} \tag{136}$$

Taking the limit on both sides as t gets large, and using equation (135), one finds

$$\lim_{t \to \infty} \mathbb{E}^* \left[\mathbb{1} \left\{ \int_0^t \alpha_{st} d\xi_s > B(t, \omega t) \right\} \right] = 1. \tag{137}$$

To proceed, recall from (38) and (119) that in order to show that the pricing kernel is not UI when $R_\infty = 0$ one needs to prove that

$$\lim_{\delta \to \infty} \sup_t \left(P_{0t} \mathbb{E}^* \left[\mathbb{1} \left\{ \int_0^t \alpha_{st} d\xi_s > B(t, \delta) \right\} \right] \right) > 0. \tag{138}$$

To see that (138) holds, note that

$$\begin{aligned}
\lim_{\delta \to \infty} \sup_t &\left(P_{0t} \mathbb{E}^* \left[\mathbb{1} \left\{ \int_0^t \alpha_{st} d\xi_s > B(t, \delta) \right\} \right] \right) \\
&= \lim_{T \to \infty} \sup_t \left(P_{0t} \mathbb{E}^* \left[\mathbb{1} \left\{ \int_0^t \alpha_{st} d\xi_s > B(t, \omega T) \right\} \right] \right) \\
&= \limsup_{T \to \infty} \sup_t \left(P_{0t} \mathbb{E}^* \left[\mathbb{1} \left\{ \int_0^t \alpha_{st} d\xi_s > B(t, \omega T) \right\} \right] \right) \\
&\geq \limsup_{T \to \infty} P_{0T} \mathbb{E}^* \left[\mathbb{1} \left\{ \int_0^T \alpha_{sT} d\xi_s > B(T, \omega T) \right\} \right] \\
&= \limsup_{T \to \infty} P_{0T}, \tag{139}
\end{aligned}$$

where the final equality follows as a consequence of equation (137). Keeping in mind that for the case under consideration we have $R_\infty = 0$, it follows from (95) and (98) that P_{0T} is of the form

$$P_{0T} = \exp \left(-\frac{1}{k}(r_0 - \theta) \left(1 - e^{-kT} \right) + f(T) \right) \tag{140}$$

for some function $f(T)$ that satisfies

$$\limsup_{T \to \infty} |f(T)| < \infty. \tag{141}$$

We deduce that

$$\limsup_{T \to \infty} P_{0T} = \exp\left(-\frac{1}{k}(r_0 - \theta)\right) \limsup_{T \to \infty} \exp\left(f(T)\right) > 0. \tag{142}$$

Therefore, the right side of (139) is strictly positive, and thus we have shown that (138) holds, which concludes the proof. $\qquad\square$

X. LONG-BOND RETURN IN LÉVY-VASICEK MODEL

We proceed to determine the return on a unit investment in the long bond in a Lévy-Vasicek model. Note that

$$\lim_{T \to \infty} \log \frac{P_{tT}}{P_{0T}} = \lim_{T \to \infty} \left[-R_{tT}(T - t) + R_{0T}T\right], \tag{143}$$

from which it follows that

$$
\begin{aligned}
L_t &= (\theta + \psi(-\lambda))t + \frac{1}{k}(r_0 - r_t) - \lim_{T \to \infty} \int_0^t \psi(\alpha_{sT})\mathrm{d}s \\
&= (\theta + \psi(-\lambda))t + \frac{1}{k}(r_0 - r_t) - \psi\left(\frac{\sigma}{k} - \lambda\right)t \\
&= R_\infty t + \frac{1}{k}(r_0 - r_t).
\end{aligned}
\tag{144}
$$

It is natural to ask whether it is possible to bring the long-bond return process into the form of a geometric price process. If we take into account relation (78) for the integrated short rate and definition (73) for the excess rate of return, then after some algebra we deduce that

$$L_t = \exp\left[\int_0^t \left(r_s + R\left(\lambda, \frac{\sigma}{k}\right)\right)\mathrm{d}s + \frac{\sigma}{k}\xi_t - \psi\left(\frac{\sigma}{k}\right)t\right]. \tag{145}$$

Thus, the form of the long-bond return is indeed that of a geometric asset price, with a Lévy-Vasicek short rate r_t, risk aversion λ, and volatility σ/k. If we multiply this expression by the pricing kernel, we obtain the geometric Lévy martingale

$$M_t = \exp\left[\left(\frac{\sigma}{k} - \lambda\right)\xi_t - \psi\left(\frac{\sigma}{k} - \lambda\right)t\right]. \tag{146}$$

This gives the result that Ross recovery holds in a Lévy-Vasicek model if and only if $\lambda = \sigma/k$, just as in the Brownian context.

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